WIENER MEASURE

1. Wiener measure is the Wiener process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 1.1. *The Wiener process starting at* 0 *is the* $\mathbb{R}^d$ *valued stochastic process* $W = (W(t))_{t\in[0,T]}$, $0 < T < \infty$ *such that*

(1) $W(0) = 0$ *a.s.;*
(2) *(ii) the family of distributions is specified by*

$$F_{t_1,\dots t_k}(A) = \int_A p(0,x,t_1,x_1)p(t_1,x_1,t_2,x_2)\dots p(t_{n1},x_{n-1},t_n,x_n)dx_1\dots dx_n$$

for every Borel set A *in* $\mathbb{R}^d \times \mathbb{R}^d$ *(*n *times).*

Here

$$p(s,x,t,y) = \frac{1}{(2\pi(t-s))^{d/2}}e^{-\frac{|x-y|^2}{2(t-s)}}.$$

Another characterization is

(1) $W(0) = 0$ a.s.;
(2) $W(t) - W(s)$ is $\mathcal{N}(0, |t-s|)$;
(3) $W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$ are independent for all $0 \le t_1 < t_2 \dots < t_n$;
(4) $t \mapsto W(t)$ is continuous a.e. $\omega \in \Omega$.

Broadly speaking, a stochastic process can be looked many ways. For example, has a:

(1) probability measure (`set of functions`, `convenient` σ `algebra`, `probability`);
(2) family of random variables $X(t) : \Omega \to \mathbb{R}^d$, $t \in [0,T]$;
(3) random element $X : \Omega \to$ `set of functions`;
(4) family of distributions $\mu_{t_1,\dots,t_n}$ in $(\mathbb{R}^d)^n$.

The Kolmogorov Extension Theorem makes the last description above equivalent to the other descriptions and it gives meaning to the starting definition: there exists such a process, because it does lead to a unique probability measure on a convenient probability space.

1.1. Existence.

We will need the following consistency notion. Let

$$\{t_1,\dots,t_m\} \subset \{s_1,\dots,s_n\} \subset [0,T], \quad m < n$$

and

$$s_1^{(0)} < \dots < s_{n_0}^{(0)} < \mathbf{t}_1 < s_1^{(1)} < \dots < s_{n_1}^{(1)} < \mathbf{t}_2 < \dots < \mathbf{t}_m < s_1^{(m)} < \dots < s_{n_m}^{(m)}.$$

Note: the previous, just means we can fit, n_0 “s" before t_1, n_1 “s" between t_1 and t_2 and so forth; we used negrito to emphasize the positions.

We say a family of distributions $\mu_{t_1,\dots,t_m}$ is consistent if

$$\mu_{t_1,\dots,t_n}(E_1\times\dots\times E_m) = \mu_{s_1,\dots,s_n}(\mathbb{R}^d\times\dots\times\mathbb{R}^d\times\mathbf{E}_1\times\mathbb{R}^d\times\dots\mathbb{R}^d\times\ \mathbf{E}_m\times\mathbb{R}^d\times\dots\times\mathbb{R}^d),$$

for all Borel sets E_1, $E_2, \dots$ of $\mathbb{R}^4$. Note: again the negrito is just for visualization.

(1) K**olmogorov extension theorem**: let $\mu_{t_1,\dots t_k}$, be a family of probability measures satisfying the consistency condition. Then there exists a measurable space $(\Omega_K, \mathcal{B}_K)$, a unique probability measure P and a stochastic process X such that the

$$P(\omega : (X(t_1,\omega),\dots,X(t_n,\omega)) \in A) = \mu_{t_1,\dots t_n}(A),\ \text{for } A \in (\mathbb{R}^d)^n.$$

(2) **Kolmogorov continuity criteria**. Let $(X(t))_{[0,T]}$ be a stochastic process and assume there exists $p, \beta, C > 0$

$$\mathbb{E}|X(t) - X(s)|^p \leq C|t - x|^{1+\beta}, \; x, y \in [0, T].$$

Then X has γ Holder continuous modification for all $0 < \gamma < \beta/p$.

As for continuity, since $W(t) - W(s) = \sqrt{|t - s|}\xi$ and ξ is $\mathcal{N}(0, 1)$, we have

$$\mathbb{E}|W(t) - W(s)|^p = E|\xi|^p |t - s|^{p/2}.$$

Hence there exist a modification in the Hoelder spaces of order $\gamma < 1/2 - 1/p$, for all p. Letting $p \to \infty$, there exists a continuous modification in the Holder spaces of order $\gamma < 1/2$.

To finalize, look at the family $F_{t_1,\ldots t_k}$. It can be shown that it is a consistent family. Kolmogorov extension enters and the construction of the theorem yields:

(1) $\Omega_K = (\mathbb{R}^d)^{[0,T]}$;
(2) $\mathcal{B}_K$ the smallest $\sigma-$ algebra making measurable the evaluation maps $\bar{t} : \omega \mapsto \omega(t)$ (the point of this is so that the process built $W(t)$ is measurable for all t);
(3) the random variable is $W(t)$ is the th projection or evaluation map $\omega \mapsto \bar{t}(\omega) = \omega(t)$;
(4) $\mathbb{P} = \mathbb{P}_W$ is such that $\mathbb{P}_W(\omega : W(t_1, \omega), \ldots, W(t_n, \omega)) \in A) = F_{t_1,\ldots t_k}(A)$, A Borel of $(\mathbb{R}^d)^n$.

In some sense the original probability space is not relevant anymore. We can just assume the Wiener process is the constructed (canonical) process $\Omega_K \times [0, T] \to \mathbb{R}^d$, $(t, \omega) \mapsto \bar{t}(\omega) = \omega(t)$ and that the probability space is $(\Omega_K, \mathcal{B}_K, \mathbb{P}_W)$. Normally we drop the the subscripts as well.

On the other hand, one can argue that the original space does matter after all. More about this just bellow.

2. Discussion

2.1. **Nuisances.** As it stands upt to this point

(1) Ω_K is too big, or
(2) $\mathcal{B}_K$ is too small.

In fact,the continuous functions are not in $\mathcal{B}_K$, essentially because to determine a continuous function it is not sufficient to know what happens in a finite (or countable) points of its domaint. See Durrett, Ex. 8.1.1.

How to make sense of the existence of continuous modification a.s. then? Should we not have $\mathbb{P}(C[0, T]) = 1$ in some sense? This has been answered. We sketch here the answer:

(1) show $P(\Omega_D) = 1$ where

$$\Omega_D = \{F : D \to (\mathbb{R}^d), \text{F is unformily continuous}\} \subset \text{C[0, T]}, \quad \text{D countable dense in[0, T]};$$

(2) $\Omega_D \subset \Omega$, so it induces (lifts to) a probability measure $\overline{P}$ on $\Omega = C[0, T]$;
(3) show $\overline{P}$ has the associated distributions $F_{t_1,\ldots t_k}$.

$\overline{P}$ is the Wiener measure and $\Omega = C[0, T]$ is called the Wiener space.

See [Var80, Section 4] or [Dur96, Theorem 8.1.12] for this.

3. Food for thought

It was suggested that we can construct the Wiener measure directly on $C[0, T]$, by carrying out the proof with another σ algebra introducing a topology first and taking the Borel σ algebra afterwards. We can go with:

(1) the topology τ_p generated by all evaluation maps (this is the smallest topology that renders evaluation maps continuous) on Ω_K;
(2) the topology τ_u generated by all linear functionals (not only the evaluation maps).

In our understanding, this is still problematic: we are adding more restrictions to the generating set of the $\sigma-$ algebra, so we think that the final configuration should observe $\sigma(\tau_u) \subset \sigma(\tau_p) \subset \mathcal{B}_K$, a priori. Are we making the problem worst? Recall that the problem was that $\mathcal{B}_K$ was too small as it did not even contain $C[0,T]$. A more careful analysis is needed.

In any case, historically Wiener is supposed to have built Wiener process directly in $C[0,1]$: how was that construction carried out?

4. Quasi-invariance

The Wiener space $C[0,T]$ is a separable Banach space under the uniform norm. Separability comes from approximation by Bernestein polynomials of rational coefficients, which form a countable subspace of the Wiener space. See a proof based on LLN in [Dur96, Example 2.2.1].

The Wiener measure is quasi-invariant for translations. Let γ be the Wiener measure.

Theorem 4.1 (Cameron-Martin). *$\gamma_h = \gamma(\cdot - h)$ is absolutely continuous wrt to γ iff $h \in H^1_0[0,T]$, where*

$$H^1_0[0,T] = \{\varphi \text{ absolutely continuous on } [0,T], \varphi(0) = 0, \dot{\varphi} \in L_2(0,T)\}.$$

See [Hai09, Theorem 3.41] here.

$H^1_0[0,T]$ is called the Cameron-Martin space. It is a Hilbert space under the scalar product

$$(h,g)_{H_\gamma} = (h,g)_\gamma = \int_0^T \dot{h}(s)\dot{g}(s)ds,$$

and there is Sobolev embedding $H^1_0[0,T] \simeq W^{1,2}(0,T) \hookrightarrow C[0,T]$ (see [AF03, Theorem 4.12]).

At the seminar there has been a discussion about measure on groups and invariance of the measure by the group operation.

In some sense the Wiener measure is quasi-invariant for the additive group of translations. Note that the "quasi" takes out almost too many things: first, it is not the case that the measures are invariant, but rather that they give the same null sets; secondly, the only allowed translations are the ones in the space $H^1_0[0,T]$, other than that the measures are singular; and, at last, it can be shown that this space is very "small" $\gamma(H^1_0[0,T]) = 0$. See [Hai09, section 3.2] here.

In general, this is the common situation in infinite dimensional spaces: there is no locally finite and translation invariant measure in separable Banach spaces. See a short proof at Wiki.

This is almost the opposite situation of that of the finite dimensional case. To see this: take a non degenerate Gaussian measure in R^d, then the covariance matrix C is definite-positive and the Cameron-Martins space is $\sqrt{C}\mathbb{R}^d \simeq \mathbb{R}^d$: every direction gives an equivalent measure. For degenerate measures we are just moving the point measure around, so we get the same.

5. "Abstract non-sense" and deviations

The Wiener measure can be considered as a particular case of Gaussian measure on a separable Banach space. This is extensively studied in[Bog98, Bax76] but we only make the following closing remarks.

An abstract Wiener space is a triple (i, H, E) where E is separable Banach space and H is a Hilbert space continuously embedded in E. The injection i is the embedding. This definition abstracts the situation encountered for the Wiener measure, discussed above: $E = C[0,T]$, $H = H^1_0[0,T]$ and i is the inclusion map.

In general, given a Gaussian measure in a separable Banach space E, the asymptotic behavior of a $\mu_\varepsilon = \mu(\frac{\cdot}{\varepsilon})$, as ε vanishes is described using the above construction. For sake of interpretation, think of X as an $E-$valued zero-mean Gaussian element. And think in terms of evolution: as ε vanishes what happens to the laws $\mu_\varepsilon(A) = \mathbb{P}(\varepsilon X \in A)$, A Borel of E? The law of εX becomes more concentrated at 0, s ε vanishes. This becomes encoded in a property called the large deviations principle. Loosely speaking, the next function encodes how "how much probability mass there is around a point" $x \in E$, or "how much energy is required to be at" x. The rate function is

$$I(x) = \begin{cases} \frac{1}{2}|x|_\mu^2, \ x \in H_\mu \\ \infty, \text{otherwise} \end{cases}$$

where $(H_\mu, (\cdot, \cdot)_\mu)$ is the Cameron-Martin space associated to μ the Gaussian law of X: it is embedded in E. Hence we obtain another example of an abstract Wiener space: (i, H_μ, E). See [DPZ92, Section 12.1.2].

References

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