First, Rado introduced three results about empirical distribution

$$\hat{F_n}(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{(X_i \le t)}.$$

1. **Strong Law of Large Numbers.** For all t,
$$\hat{F_n}(t) \xrightarrow{\text{a.s.}} F(t)$$

2. **Glivenko-Cantelli.**
$$\left\|\hat{F_n} - F\right\|_\infty := \sup_{t\in\mathbb{R}} \left|\hat{F_n}(t) - F(t)\right| \xrightarrow{\text{a.s.}} 0.$$

3. **Dvoretzky-Kiefer-Wolfowitz.** For all $\epsilon > 0$,
$$P\left(\sup_{i\in\mathbb{N}} \left|\hat{F_n}(i) - F(i)\right| > \epsilon\right) \le 2e^{-2n\epsilon^2}.$$

Ideally, we want to replace $\hat{F}$ with the empirical distribution of the eigenvalues and F with the semicircular distribution.

Next, we consider Haar measures. Before we define a Haar measure, we recall what a topological group is. A *topological group* G is a topological space and group such that the group operations of product

$$G \times G \to G : (x, y) \mapsto xy$$

and taking inverses

$$G \to G : x \mapsto x^{-1}$$

are continuous functions. Now, take $(G, .)$ be a locally compact, i.e. every point has a compact neighborhood, Hausdorff topological group. Define the *left* (resp. *right*) *translate of* G as $gS = \{ g.s : s \in S \}$ (resp. $Sg = \{ s.g : s \in S \}$). A measure μ is said to be *left translation invariant* if for all Borel subsets $S \subseteq G$ and for all $g \in G$, one has $\mu(gS) = \mu(S)$. Right translation invariant measures are similarly defined.

Haar proved a theorem stating that there exists, up to a positive multiplicative constant, a unique countably additive, nontrivial measure μ on the Borel subsetse of G satisfying the following properties:

1. μ is left translation invariant: $\mu(gE) = \mu(E)$ for all $g \in G$ and all Borel set E.
2. μ is finite on every compact set: $\mu(K) < \infty$ for all compact K.
3. μ is outer regular on Borel sets E: $\mu(E) = \inf\{ \mu(U) : E \subseteq U, U \text{ open and Borel} \}$.
4. μ is inner regular on open Borel sets E: $\mu(E) = \sup\{ \mu(K) : K \subseteq E, K \text{ compact} \}$.

Such a measure is called a *left Haar measure*. Using the general theory of Lebesgue integration, we define an integral for all Borel measurable functions f on G, called the *Haar integral*:

$$\int_G f(sx)\, d\mu(x) = \int_G f(x)\, d\mu(x).$$

Some examples of Haar measures (including the ones covered in class) are:

1. A Haar measure on $(\mathbb{R}, +)$ which takes the value 1 on $[0, 1]$ is the restriction of lebesgue measure to the Borel subsets of $\mathbb{R}$. This can be generalized to $(\mathbb{R}^n, +)$.
2. If $G = (\mathbb{R} \setminus \{ 0 \}, \times)$, then
$$\mu(S) = \int_S \frac{1}{|t|}\, dt$$
for any Borel subset S, is a Haar measure on G.

3. If $G = GL(n, \mathbb{R})$, any left Haar measure is a right Haar measure and one such measure is given by

$$\mu(S) = \int_S \frac{1}{|\det(X)|^n} \, dX,$$

where dX denotes the Lebesgue measure on $\mathbb{R}^{n^2}$.

4. On the unit circle T, consider the function $f : [0, 2\pi] \to T$ defined by $f(t) = (\cos(t), \sin(t))$. Then, μ defined by

$$\mu(S) \frac{1}{2\pi} m \left(f^{-1}(S)\right),$$

where m is the Lebesgues measure, is a Haar measure.

We also discussed group actions in order to better understand the translation invariance in Haar integrals. Given a group G and a set X, a *(left) group action* of G on X is a function $G \times X \to X$, $(g, x) \mapsto g.x$ satisfying the following two axioms:

1. $(gh).x = g.(h.x)$ for all $g, h \in G$ and all $x \in X$.
2. $1_G.x = x$ for all $x \in X$.

Another way to think of a group action is to consider it as a group homomorphism from G into $\mathrm{Sym}(X)$, the symmetric group of all bijections from X to X. After a long debate, we did establish that

$$\int f(g) \, d\mu = \int f(a^{-1}g) \, d\mu.$$