MATH 705 4th MEETING SUMMARY

February 6th, 2013

We began by discussing some examples of ordinary differential equations (system) with random coefficients (matrix).

Example 1. We consider the equation as the form of

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where a, b, c, d are i.i.d random variables satisfying the uniform distribution $\mathbb{U}(-9,9)$. The characteristic equation is

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0.$$

Example 2.

$$y'' + ay' + by = 0,$$

where a, b are i.i.d random variables satisfying the uniform distribution $\mathbb{U}(-9,9)$ and the characteristic equation is

$$\lambda^2 + a\lambda + b = 0.$$

Take the Example 2 for instance, we can classify the behavior of the solution based on values of (a,b). And for certain random variables a and b, we can futher study the probability or distribution of the solution taking certain structure. We can also study the structure stability for a particular class of solutions.

Afterwards, John gave a presentation about interesting results for random matrices.

John started with a result of the emprical distribution of the bulk of eigenvalues of covariance matrices, found by Marienko-Pastur. Suppose $X \in M_{n,p}(\mathbb{R})$, with rows independent Gaussian $\mathbb{N}(0,\Sigma)$, where Σ is an identity matrix and all entries of X are i.i.d satisfyin g the nomal distribution $N(0,1)$.

The following result is about the study of singular values of X, or eigenvalues of $W := X^T X$.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_p$ be eigenvalues of W. Define

$$G_p(t) = \frac{1}{p} \#\{\lambda_i \leq nt\}.$$

If $n, \ p \to \infty$, such that $\frac{n}{p} \to \gamma \geq 1$, then

$$G_p(t) \to G(t), \ \text{a.s.},$$

where

$$G'(t) = g(t) = \frac{\gamma}{2\pi t} \sqrt{(b-t)(t-a)} \cdot 1_{[a,b]}(t),$$

$$a = (1 - \gamma^{-1/2})^2, \quad b = (1 + \gamma^{-1/2})^2.$$

Then we spent some time discussing the concrete example with $\lambda = 1$ in an effort to get some intuitive understanding of the above result.

Johansson and Johnstone derived the fluctuation theory of the extreme eigenvalues for the Gaussian case. The limiting distribution fo dominant eigenvalue of $n \times n$ Gaussian symmetric matrix is given by

$$F_1(s) = \exp\left(-\frac{1}{2} \int_s^\infty q(x) + (x-s)q^2(x)dx\right), \ \text{(Tracy-Widom Distribution)},$$

where q solves the Painlevé II Equation

$$q''(x) = xq(x) + 2q^3(x)$$

with b.c. $q(x) \sim A_i(x)$ as $x \to \infty$, and

$$A_i(x) = \frac{1}{x} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt,$$

where $A_i(x)$ solves $y''(x) = xy(x)$.