Math 705 Meeting Summary of 3rd Session (January 30, 2013)
(Summary by Albert)

Prof. Lototsky began the session with an exploratory question about the statement of the Semi-Circle Law: What kind of information do we get about the eigenvalues? The observation here is that the object in the theorem is not the same as the typical empirical measure, even though some numerical experiments suggest that the behaviors are similar.

Radoslav presented the following:

Given Z_{ij}, Y_i - two families of independent random variables, each family iid, with Z_{ij} mean zero and variance 1, along with $\max(E|Z_{ij}|^k, E|Y_i|^k) < \infty \ \forall k \in \mathbb{N}$.

Construct Wigner matrix with Ys on main diagonal and Zs off the main diagonal subject to symmetry, scaled by $\frac{1}{\sqrt{N}}$:

$$\frac{1}{\sqrt{N}} \begin{bmatrix} Y_1 & Z_{12} & Z_{13} & \cdots \\ Z_{12} & Y_2 & Z_{23} & \cdots \\ Z_{13} & Z_{23} & Y_3 & \cdots \\ \vdots & & & \ddots & \\ & & & & Y_N \end{bmatrix}$$

Order the eigenvalues λ_i^N: $\lambda_1^N \leqslant \lambda_2^N \leqslant \cdots \leqslant \lambda_N^N$.

Then $L_N := \frac{1}{N} \sum_i \delta_{(\lambda_i^N)} \xrightarrow{P} G(x) := \frac{1}{2\pi} \sqrt{4-x^2}\, \mathbb{1}_{|x| \leqslant 2}$.

Lototsky's challenge question:

What can we say about the relationship, if any, between the L_n, as defined in the foregoing, and the emperical distribution $F_N := \frac{1}{N} \sum \delta_{X_i}$ for a series of actualizations $X_1, X_2, \ldots, X_n, \ldots$?

Lototsky: $L_N = \frac{1}{N} \sum_i \delta_{(\lambda_i^N)} \overset{?}{\longleftrightarrow} \lim_{M \to \infty} \frac{1}{M} \sum_{k=1}^{M} L_N^{(k)} \overset{?}{=} \mathcal{L}aw(L_N)$

Radoslav agreed to and will make/simulate histograms of eigenvalues considered in the foregoing, as well as L_N for a large fixed N.

Haininy's presentation on Classical Matrix Groups:

<u>Skew Field</u>: Set K with operation "+" and "$\cdot$", having the property: $a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \forall a, b, c \in K$.

K is abelian group under $+$. Take 0 to denote the "additive" identity ($0 + a = a \ \forall a \in K$). Then $K - \{0\}$ is a group under the operation $\cdot$. Examples: $\mathbb{R}$ and $\mathbb{C}$ are fields, whereas $\mathbb{R}$, $\mathbb{R}^2 \sim \mathbb{C}$, $\mathbb{R}^4 \sim \mathbb{H}$ are skew fields.

$f : \mathbb{C} \to \mathbb{R}^2$ defined by $f(a+ib) = (a,b)$,

$g : \mathbb{H} \to \mathbb{C}^2$ defined by $g(a+bi+cj+dk) = ((a+bi),(c+di))$

Note $a+bi+cj+dk = (a+bi) + (c+di)j$ since $ij = k$.

Let $\mathbb{K}$ denote a skew field.

$GL_n(\mathbb{K}) := \{A \in M_n(\mathbb{K}) : A \text{ is invertible}\} = \{A \in M_n(\mathbb{K}) : \det(A) \neq 0\}$

(Recall: $A \in M_n(\mathbb{K})$ is invertible if $\exists B \in M_n(\mathbb{K})$ st $AB = BA = I$).

$f : \mathbb{C}^n \to \mathbb{R}^{2n}$,

$<x,y>_{\mathbb{C}} = <f(x), f(y)>_{\mathbb{R}} + i <f(x), f(iy)>_{\mathbb{R}} \in \mathbb{C}$,

$h := f_{2n} \circ g_n : \mathbb{H}^n \to \mathbb{R}^{2n}$,

$$\left.\begin{aligned} <x,y>_{\mathbb{H}} = <h(x), h(y)>_{\mathbb{R}} + i <h(x), h(iy)>_{\mathbb{R}} + \\ + j <h(x), h(jy)>_{\mathbb{R}} + k <h(x), h(ky)>_{\mathbb{R}} \end{aligned}\right\} \in \mathbb{H}.$$

$SL_n(\mathbb{K}) := \{A \in GL_n(\mathbb{K}) : \det(A) = 1\}$

$O_n(\mathbb{K}) := \{A \in GL_n(\mathbb{K}) : <xA, yA> = <x,y> \ \forall x,y \in \mathbb{K}^n\}$.

$O(n)$ – orthogonal group when $\mathbb{K} = \mathbb{R}$, $U(n)$ – unitary group when $\mathbb{K} = \mathbb{C}$,
$S_p(n)$ – special group when $\mathbb{K} = \mathbb{H}$.

Thm: $\{A \in GL_n(\mathbb{K}) : |\det(A)| = 1\} = O_n(\mathbb{K})$.