M705 Spring 2013 Summary for Week 2

1 Intro to RMT (Gene)

(Also see the Anderson - Guionnet - Zeitouni book, pp.6-11(?))

We start with two independent families of R.V.s, $\{Z_{i,j}\}_{1\le i<j\le n}$ and $\{Y_i\}_{1\le i\le n}$. The Z's are mean-zero and variance-one. We also impose the condition $r_k = \max(E|Z_{i,j}|^k, E|Y_i|^k) < \infty$.

To construct the Wigner Matrix, use the Y's as diagonals and place the Z's away from the diagonal with the stipulation of symmetry. Scale the entire resulting matrix by $1/\sqrt{N}$.

We are interested in the properties of the eigenvalues of the matrix. For real Y's and Z's, the eigenvalues are also real. Let the eigenvalues be λ_i^N, and order them in increasing order: $\lambda_1^N \le \lambda_2^N \le ... \le \lambda_N^N$.

Let the *empirical measure* L_N be:

$$L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i^N} \tag{1}$$

And let the *semicircle distribution* $\sigma(x)dx$ on $\mathbb{R}$ be:

$$\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{I}_{|x|\le 2} \tag{2}$$

Wigner showed that: *(Wigner semicircle theorem)* L_N converges weakly, in prob., to the semicircle distribution; i.e.

$$\lim_{n\to\infty} P(|<L_N, f> - <\sigma, f>| > \epsilon) = 0$$

for all $f \in C_b(\mathbb{R})$ and all $\epsilon > 0$.

The proof of the theorem employs the *Method of Moments*.

Define $m_k L =< \sigma, x^k >$.

It can be shown that in the given context, $m_{2k} = c_k$ and $m_{2k+1} = 0$, where c_k are the *Catalan numbers* (named after a 19th century Belgian mathematician) with $c_k = \frac{(2k)!}{(k+1)!k!}$.

Follow the statements of three lemmas.

<u>Lemma 1</u>: $c_k \le 4^k$.

<u>Lemma 2</u>: Let $\overline{L_N} = EL_N$. Then $< \overline{L_N}, f >= E < L_N, f >$. Also let $m_k^N L =< \overline{L_N}, x^k$. The Lemma states that for all $k \in \mathbb{N}$, we have $\lim_{N\to\infty} m_k^N = m_k$.

<u>Lemma 3</u>: For all $k \in N$ and $\epsilon > 0$,

$$\lim_{N\to\infty} P(|<L_N, x^k> - <\overline{L_N}, x^k>| > \epsilon) = 0.$$

We can prove Wigner's theorem using the three lemmas.

For $f \in C_b(\mathbb{R})$, we want to show that

$$\lim_{N\to\infty} <L_N, f> - <\sigma, f>$$

in probability. Via Chebyshev, we obtain

$$P(< L_N, |x|^k \mathbb{I}_{|x|>B} > \epsilon) \leq \frac{1}{\epsilon} E < L_N, |x|^k \mathbb{I}_{|x|>B} > \leq \frac{< \overline{L_N}, x^{2k} >}{\epsilon B^k}$$

By Lemmas 2 and then 1,

$$\limsup P(< L_N, |x|^k \mathbb{I}_{|x|>B} >> \epsilon) \leq \frac{< \sigma, x^{2k} >}{\epsilon B^k} \leq \frac{4^k}{\epsilon B^k}.$$

Observe that

$$\lim_{B\to\infty} \limsup_{N\to\infty} P(< L_N, |x|^k \mathbb{I}_{|x|>B} >> \epsilon) = 0.$$

Finally, for a fixed δ, we use the Weierstrass approximation theorem to produce a polynomial Q_δ sufficiently close to f, and then break $P(| < L_N, f > - < \sigma, f > | > \delta)$ into pieces:

$$\begin{aligned} P(| < L_N, f > - < \sigma, f > | > \delta) \leq\ & P(| < L_N, Q_\delta > - < \overline{L_N}, Q_\delta > | > \delta/4) + \\ & + P(| < \overline{L_N}, Q_\delta > - < \sigma, Q_\delta > | > \delta/4) + \\ & + P(| < L_N, Q_\delta \mathbb{I}_{|x|>B} > | > \delta/4) + \\ & + P(| < L_N, f \mathbb{I}_{|x|>B} > | > \delta/4) \end{aligned}$$

All terms go to 0, which completes the proof.

2 A Selected Application of RMT (Albert)

(Also see Thomas Guhr's research summary, available here:

`http://www.theo-phys.uni-essen.de/tp/ags/guhr_dir/encrmt.pdf`

For more on the Wigner surmise, see Madan Lal Mehta's book.)

The goal is to study energy correlations of quantum spectra. Supposing that the spectrum of a quantum system has been measured or calculated, all levels in the total spectrum having the same quantum numbers form one particular subspectrum. Its energy levels are at positions x_n with $n = 1, 2, ..., N$. Assume N is large.

With a proper smoothing procedure, we obtain the level density $R_1(x)$, meaning the prob.density of finding a level at the energy x. The level density increases with x for most physics systems.

However, we are interested in the spectral correlations rather than in the density. Hence we have to remove the density from the subspectrum (this is called *unfolding*). To that end we introduce a new dimensionless energy scale ξ s.t. $d\xi = R_1(x)dx$. By construction, the resulting subspectrum in ξ has level density unity. It is understood that the energy correlations are analyzed in the *unfolded subspectra*.

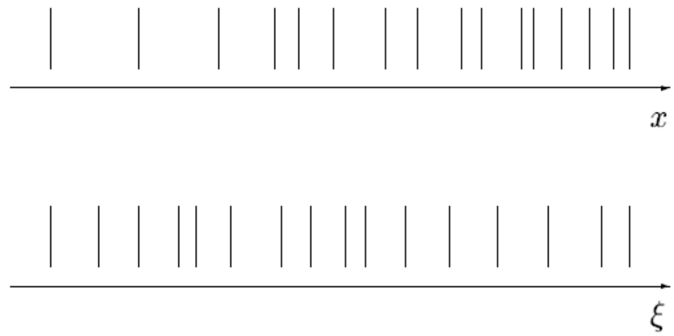

FIG. 1. Original (top) and unfolded (bottom) spectrum.

Consider the nearest neighbor spacing distribution $p(s)$, i.e. the prob. density of finding two adjacent levels in the distance s. If the positions of the levels are uncorrelated, the nnsd is exponential: $p^{(P)}(s) = \exp(-s)$. Such cases are occasionally found, but many more systems follow the *Wigner surmise* distribution: $p^{(W)}(s) = \frac{\pi}{2} s \exp(-\frac{\pi}{4} s^2)$. See figure 2.

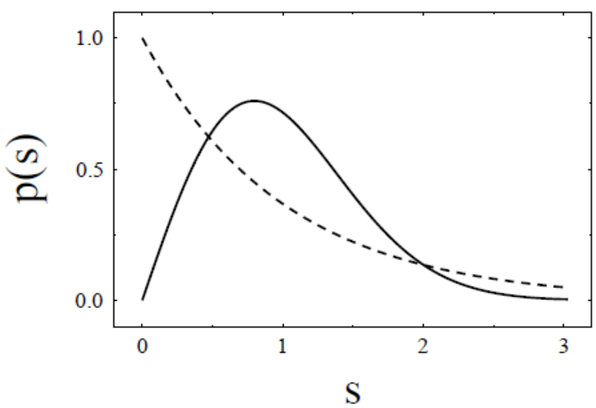

FIG. 2. Wigner surmise (solid) and Poisson law (dashed).

Does there exist a convenient mathematical model for such patterns? Yes, RMT.

To describe absence of correlations, consider a diagonal Hamiltonian $H = diag(x_1, ..., x_N)$, the eigenvalues x_n of which are uncorrelated random numbers. To model correlation, consider a full symmetric Hamiltonian with $H = H^T$ and independent random entries H_{nm}. It turns out these two models yield precisely the Exponential distribution and the Wigner surmise we saw above.

3 Types of RMs; generation of RMs (Radoslav)

3.1 (Alan Izenman: Introduction to RMT document)

Dyson 1962 classifies RM into three types based upon time-reversal invariance: 1) complex (non-time-reversal-invariant); 2) real (time-reversal invariant); 3) self-dual quaternion (time-reversal invariant with a restriction).

RM theory used in number theory, combinatorics, wireless communications, and multivariate statistical analysis and principal component analysis.

An *ensemble* of random matrices is a family/ collection of RM with a prob. density p that shows how likely a member of the family is to be observed.

Wigner and Dyson used $n \times n$ Hermitian matrices H_n with density $e^{-\beta tr[V(H_n)]}$ where V is some function of H_n, e.g. a finite polynomial function of H_n with even highest power and positive leading coefficient and constant of proportionality dependent only on n. E.g.: $V(H_n) = aH_n^2 + bH_n + c$, where a, b, c are real and $a > 0$. The entries of H_n can be real ($\beta = 1$), complex ($\beta = 2$), or real-quaternion ($\beta = 4$).

"Time-reversal" transformation: $H_n \to UH_nU^{-1}$, where U is orthogonal ($\beta = 1$), unitary ($\beta = 2$), or symplectic ($\beta = 4$). Time-reversal invariance implies that the time-reversal transformation leaves the density of H_n invariant.

And so the three Gaussian ensembles are: 1) *Gaussian orthogonal ensemble* (GOE), 2) *Gaussian unitary ensemble* (GUE), and 3) *Gaussian symplectic ensemble* (GSE).

Case 1: A is an $(n \times n)$ matrix with iid standard normal entries. H_n can be produced via $H_n = (A + A^\tau)/2$ where A^τ is the transpose of A. The diagonal entries of H_n are standard normal iids, and the off-diagonal entries are iid (up to symmetry) normals with mean-0 variance-a-half.

Case 2: A now has complex-valued iid standard-complex-normal entries. We form H_n via $H_n = (A + A^*)/2$, where A^* is the Hermitian transpose of A. The diagonal entries of H_n are iid standard normals and the off-diagonal entries are iid (up to Hermitian property) bi-normal mean-0 variance-a-half.

Case 3: Now A's entries are real-quarternian iid from $N^Q(0, 1)$. We have $H_n = (A+A^D)/2$ where A^D is the dual transpose of A.

For GOE, U is orthogonal with real entries, for GUE U is unitary with complex entries, and for GSE U is symplectic with self-dual qaternion entries. GOE is used in quantum mechanics for time-reversal invariance.

The Wigner Matrix is a real symmetric $n \times n$ matrix H_n with diagonal entries 0 and off-diagonal entries uniform ± 1. The GOE matrix is also called Winger Matrix, since it extends the original Wigner Matrix. In general, one is not restricted to Gaussian entries - other distributions also work, as long as one has the same independence and the same variance as with the GOE.

3.2 (Edelman-Rao RMT document)

Most well-studied RM have names such as: Gaussian, Wishart, MANOVA, and circular. Or Hermite, Laguerre, Jacobi and Fourier.

Hermite Ensemble – symmetric eigenvalue decomposition – $e^{-x^2/2}$ weight function – semi-circular law (W'58).

Laguerre Ensemble – singular value decomposition – $x^a e^{-x}$ weight function – Marcenko and Pastur '67.

Jacobi Ensemble – generalized SVD – $(1 - x)^a (1 + x)^b$ w.f. – generalized McKay law.

Fourier Ensemble – unitary eigenvalue decomposition – $e^{j\theta}$ – uniform.

In multivariate statistics people are interested in random covariance matrices (Wishart matrices). The construction here is: $A'A$ where A is $G_\beta(m, n)$ and A' denotes A^T, A^H or A^D depending on whether A is real, complex or quaternion.

Manova Ensemble: symmetric/ Hermitian/ self-dual $n \times n$ matrix, given by $A/(A+B)$, where A and B are $W_\beta(m_1, n)$ and $W_\beta(m_2, n)$ respectively.

Circular Ensembles: Given by $U^T U$ and U for $\beta = 1, 2$ respectively, with U a uniformly distributed unitary matrix.

3.3 (Tao's book ch2.3)

Iid matrix ensembles - all entries in a square matrix are iid r.v.s with the same distribution. Bernoulli, real gaussian, and complex gaussian ensembles defined implicitly.

Symmetric Wigner matrix ensembles defined as above.

Hermitian Wigner as above.

3.4 (Anderson-Guionnet-Zeitouni book p.6)

Construction of Wigner matrix:

Start with two independent families of iid mean-zero, real-valued r.v.s $\{Z_{i,j}\}_{1 \le i < j}$ and $\{Y_i\}_{1 \le i}$ s.t. $EZ_{1,2}^2 = 1$ and for all natural k, $r_k := \max(E|Z_{1,2}|^k, E|Y_1|^k) < \infty$.

The symmetric $N \times N$ matrix X_N with entries $Z_{i,j}/\sqrt{N}$ and $Z_{j,i}/\sqrt{N}$ away from the main diagonals, and $Y_i/\sqrt{N}$ on the main diagonal is called a *Wigner matrix*. If the r.v.s are Gaussian, the matrix is a *Gaussian Wigner matrix*.

3.5 (Some sources)

1. *Random matrix theory* by Alan Edelman and N. Raj Rao, *Acta Numerica* (2005) pp. 1-65, Cambridge University Press

2. *Introduction to Random-Matrix Theory* by Alan J. Izenman

3. *Applications of Random Matrices in Physics* by E. Brezin, V. Kazakov, D. Serban, P. Wiegmann, and A. Zabrodin, *Proceedings of NATO Advanced Study Institute on Applications of Random Matrices in Physics* (2004), Springer

4. *An Introduction to Random Matrices* by G. Anderson, A. Guionnet, and O. Zeitouni, 2010, Cambridge University Press

5. *Random Matrices* (3rd Ed.) by Madan Lal Mehta, 2004, Elsevier

6. *Topics in random matrix theory* by Terence Tao, 2010-

7. *Spectral Analysis of Large Dimensional Random Matrices (2nd ed.)* by Zhidong Bai and Jack W. Silverstein, Springer Series in Statistics, 2010, Springer