MATH 705 1ST MEETING SUMMARY

We began by discussing Tao's "dogma" that "Probability theory is only allowed to study concepts and perform operations which are preserved with respect to extension of the underlying probability space." We say that the probability space $(\Omega', \mathcal{B}', P')$ *extends* the space $(\Omega, \mathcal{B}, P)$ if there exists a measurable surjection $\pi : \Omega' \to \Omega$ which is *probability preserving* in the sense that $P'(\pi^{-1}(E)) = P(E)$ for all $E \in \mathcal{B}$. By adhering to this prescription, one is free to inject new sources of randomness as needed.

We then discussed Tao's heirarchy of confidence for an event $E = E_n$ depending on the parameter n:

(1) E holds *surely* if it is equal to the sure event $\emptyset^C$.
(2) E holds *almost surely* if $P(E) = 1$.
(3) E holds *with overwhelming probability* if, for every $A > 0$, one has $P(E) \geq 1 - C_A n^{-A}$ for some C_A independent of n.
(4) E holds *with high probability* if, for some $C, c > 0$ independent of n, one has $P(E) \geq 1 - Cn^{-c}$.
(5) E holds *asymptotically almost surely* if it holds with probability $1 - o(1)$.

Typically, n will be the dimension of the matrices under consideration. However, there was some confusion about the role of n since the dependence was suppressed in the definitions. We tried to reconcile these notions with the standard modes of convergence for random variables:

(a) $X_n \to X$ *almost surely* if, for all $\varepsilon > 0$, $\mathbb{P}\{\lim_{n\to\infty} |X_n - X| < \varepsilon\} = 1$.
(b) $X_n \to X$ *in L^p* if $\lim_{n\to\infty} E\left[|X_n - X|^p\right] = 0$. (For $p > q \geq 1$, convergence in L^p implies convergence in L^q.)
(c) $X_n \to X$ *in probability* if, for all $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\{|X_n - X| < \varepsilon\} = 1$.
(d) $X_n \to X$ *in distribution* if $\lim_{n\to\infty} E[f(X_n)] = E[f(X)]$ for all bounded, continuous f.

The preceding list is also ordered in descending order of strength, and of course there are many other characterizations of these types of convergence and a wide variety of related results. There does not appear to be a direct correspondence between Tao's hierarchy for events and the preceding hierarchy for convergence in terms of associating the event E with the random variable 1_E. For example, we can have $1_{E_n} \to 1$ almost surely without E_n holding almost surely for any n. It seems that Tao's characterizations should be regarded as indicating the certainty of events rather than addressing rates of convergence in various senses. Nonetheless, one can still show that various measures of certainty of events imply certain types of convergence of the indicators and vice versa. In some cases this is accomplished by appealing to the Borel-Cantelli Lemmas:

BC1: If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n \text{ i.o.}) = 0$.
BC2: If the events $A_1, A_2, ...$ are independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(A_n \text{ i.o.}) = 1$.

After discussing these matters, we decided that we should begin by coming to terms with some of the fundamental concepts in random matrix theory before diving into the books *Topics in Random Matrix Theory* by Terence Tao and *An Introduction to Random Matrices* by Greg Anderson, Alice Guionnet, and Ofer Zeitouni. Accordingly, we agreed that next week will begin with the following short presentations:

Albert: Applications of random matrices
Gene: Wigner matrices
Haining: Classical Lie groups and Haar measure

Jie: Gaussian Unitary Ensembles

John: Major theorems, questions, and conjectures in random matrix theory

Radoslav: Typical distributions used in generating random matrices

Umit: Random matrices over finite fields

(Jie and Umit were not there when we made this list, so they can talk about another related topic if they so desire.)