# Summary[1] of Vandermonde.

**The identity:**

$$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k},$$

which can be seen, among other ways, by (a) comparing the coefficients in suitable binomial expansions, or (b) by establishing a bijection between two procedures for selecting $r$ objects out of $m+n$, or (c) by counting suitable paths on the planar square grid. Vandermonde published it in 1772, but the result had been around for a while, going back to Zhu Shijie (1303).

Considering binomial formulas for non-integer powers leads to the *Chu–Vandermonde* identity and then to the *Gauss hypergeometric theorem.*

**The determinant/matrix/polynomial:** for all $x_1, \ldots, x_n$,

$$V_n(x_1,\ldots,x_n) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1\le k<\ell\le n} (x_\ell - x_k).$$

The matrix can be transposed and even non-square and still go under the same name.

The proof without formulas goes like this:

(1) By the definition of the determinant, $V_n(x_1,\ldots,x_n)$ is a polynomial in $x_1,\ldots,x_n$ of degree $0+1+\ldots+(n-1) = \frac{n(n-1)}{2}$ and $V_n(x_1,\ldots,x_n)=0$ if $x_k = x_\ell$ for some $k \ne \ell$.
(2) Therefore, $V_n(x_1,\ldots,x_n)$ is divisible by $x_k - x_\ell$ for all $k \ne \ell$.
(3) On the other hand, the degree of the polynomial $\prod_{1\le k<\ell\le n}(x_\ell - x_k)$ is also $\frac{n(n-1)}{2}$, and therefore $V_n(x_1,\ldots,x_n) = c\prod_{1\le k<\ell\le n}(x_\ell - x_k)$ for some number $c$.
(4) Finally, argue that $c=1$ by comparing the coefficients of a particular monomial, such as $x_2x_3^2\cdots x_n^{n-1}$.

There are more explicit proofs using induction and elementary row/colulmn operations or *LU* factorization.

For integer $x_1=1$, $x_2=2$, $x_3=3$, etc. we get the sequence

$$V_1(1)=1,\ V_2(1,2)=1,\ V_3(1,2,3)=2,\ V_4(1,2,3,4)=12,\ V_5([1,5])=288,$$
$$V_6([1,6])=34560,\ V_7([1,7])=24883200,\ V_8([1,8])=125411328000,\ldots,$$

with $V_n([1,n]) = \prod_{k=1}^{n-1} k!$ known as (one version of) *super-factorial*; see[2] OEIS A000178.

**The person:** Alexandre-Théophile Vandermonde (1735–1796), French from Paris,

- violinist, until around 1770;
- mathematician, starting from around 1770;
- chemist, in particular, investigated the metallic layers of iron.

Beside the identity, his other work in combinatorics included study of knight tours and knots. Beside the determinant [that is nowhere to be found in his papers....], his other work in algebra included study of symmetric functions, as well as the equation $x^n = 1$ and the resulting *cyclotomic polynomials.*

[1]Sergey Lototsky, USC; version of May 28, 2023.
[2]https://oeis.org/A000178