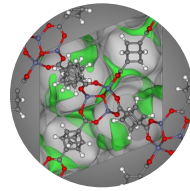

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MATHEMATICS

Generating uniformly distributed numbers on a sphere

BY CORY SIMON

📅 FEBRUARY 27, 2015

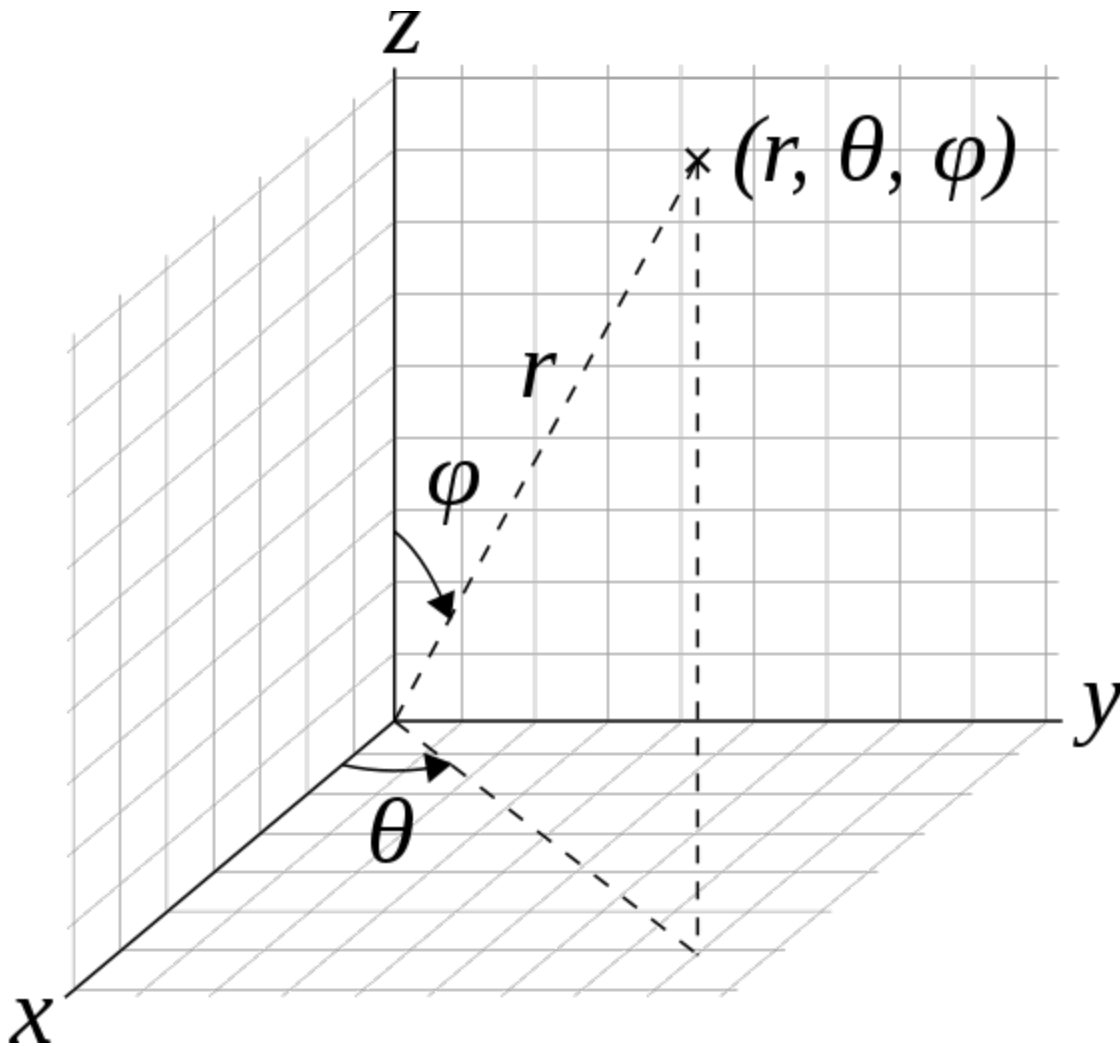
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So, we want to generate uniformly distributed random numbers on a unit sphere. This came up today in writing a code for molecular simulations. Spherical coordinates give us a nice way to ensure that a point (x, y, z) is on the sphere for any (θ, ϕ) :



$$x = r \sin(\phi) \cos(\theta)$$

$$y = r \sin(\phi) \sin(\theta)$$

$$z = r \cos(\phi).$$

In spherical coordinates, r is the radius, $\theta \in [0, 2\pi]$ is the azimuthal angle, and $\phi \in [0, \pi]$ is the polar angle.

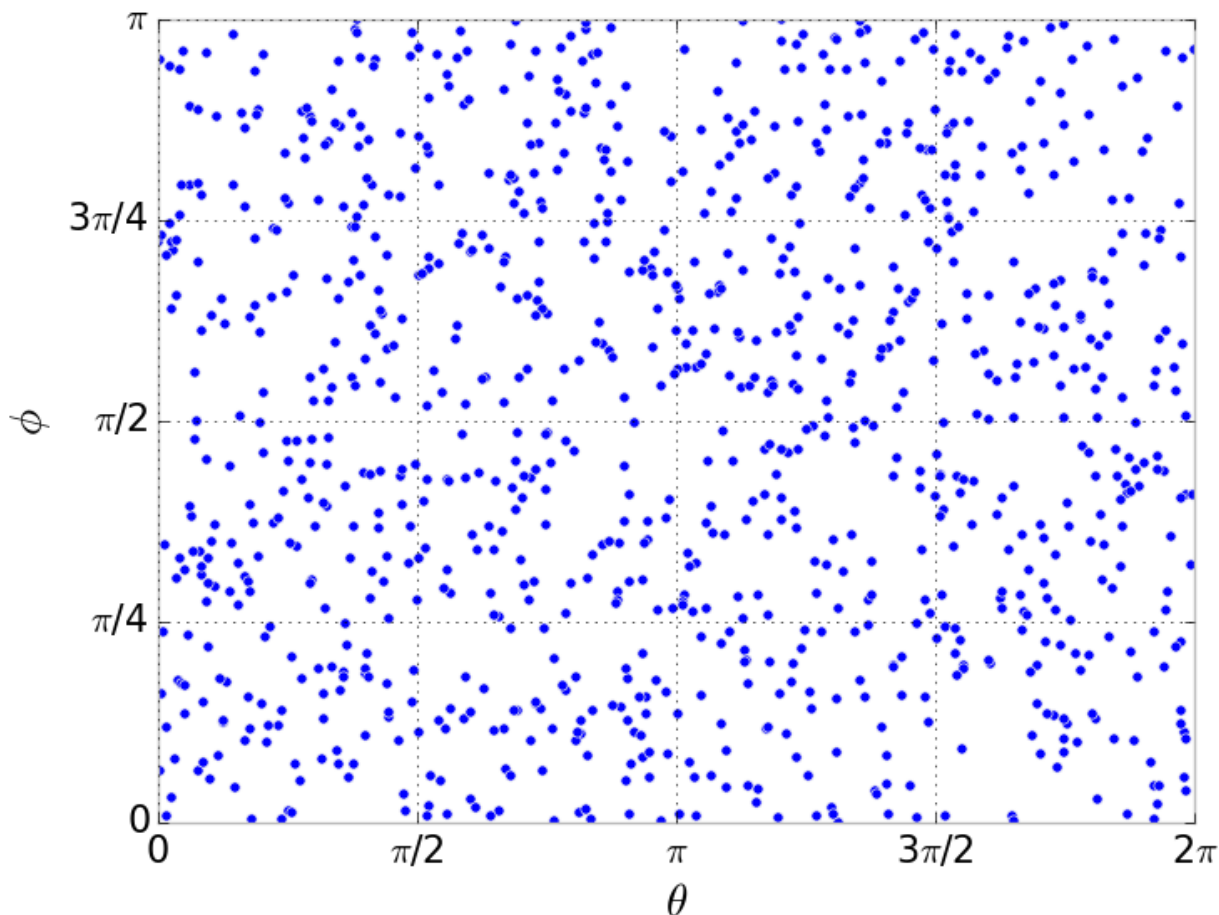
A tempting way to generate uniformly distributed numbers in a sphere is to generate a uniform distribution of θ and ϕ , then apply the above transformation to yield points in Cartesian space (x, y, z) , as with the following C++ code.

```
#include<random>  
#include<cmath>
```

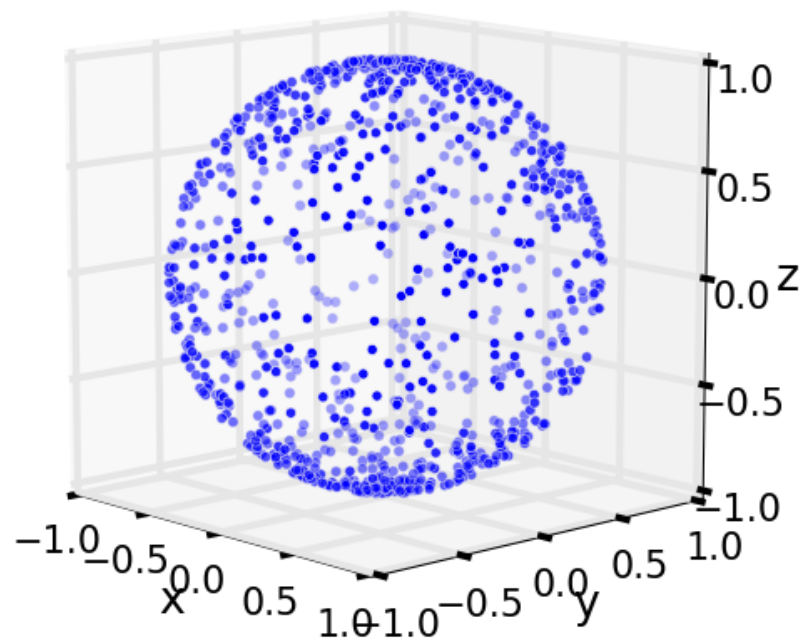
```
#include<chrono>
```

```
int main(int argc, char *argv[]) {  
    // Set up random number generators  
    unsigned seed = std::chrono::system_clock::now().time_since_epoch().count(  
    std::mt19937 generator (seed);  
    std::uniform_real_distribution<double> uniform01(0.0, 1.0);  
  
    // generate N random numbers  
    int N = 1000;  
  
    // the incorrect way  
    FILE * incorrect;  
    incorrect = fopen("incorrect.csv", "w");  
    fprintf(incorrect, "Theta,Phi,x,y,z\n");  
    for (int i = 0; i < N; i++) {  
        // incorrect way  
        double theta = 2 * M_PI * uniform01(generator);  
        double phi = M_PI * uniform01(generator);  
        double x = sin(phi) * cos(theta);  
        double y = sin(phi) * sin(theta);  
        double z = cos(phi);  
        fprintf(incorrect, "%f,%f,%f,%f,%f\n", theta, phi, x, y, z);  
    }  
    fclose(incorrect);  
}
```

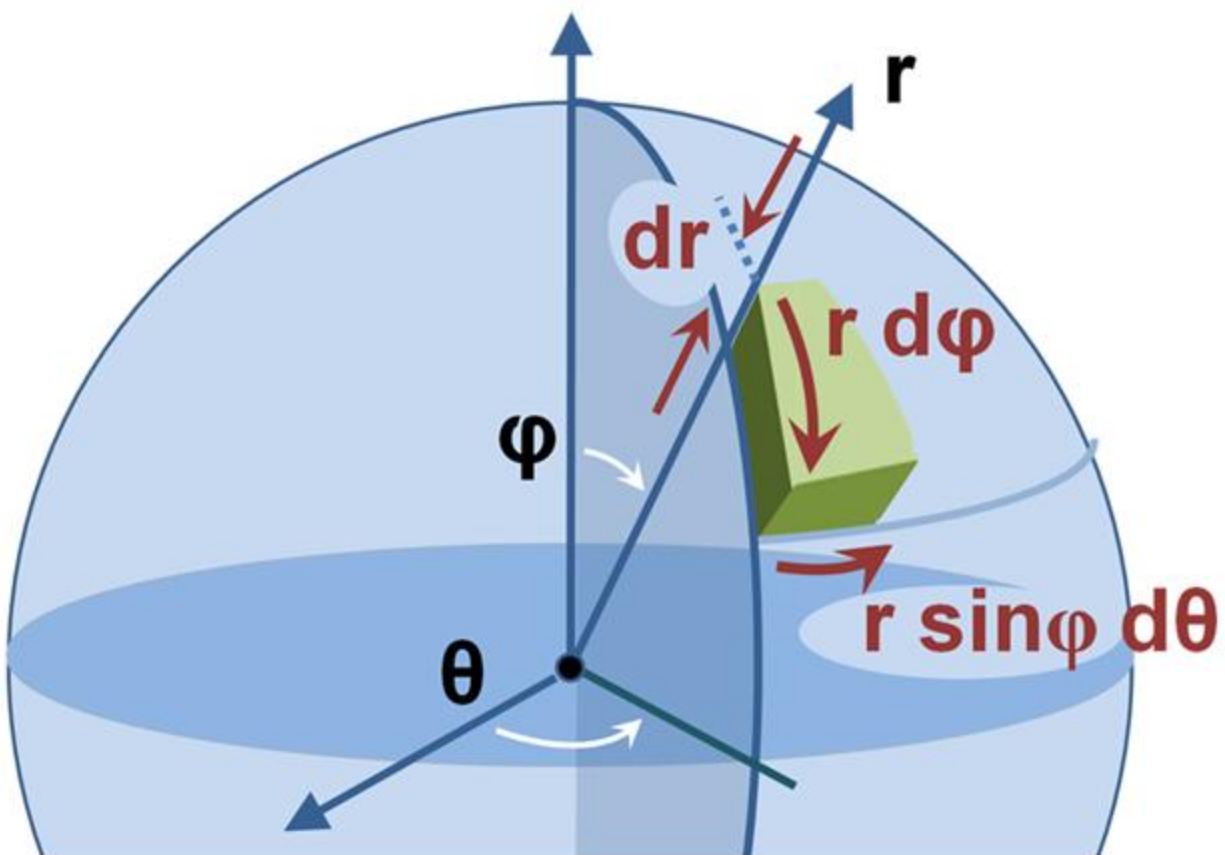
The distribution in the θ - ϕ plane in this strategy is uniform:



After mapping these points in the θ - ϕ plane to the sphere using the relationship between spherical and Cartesian coordinates above, this *incorrect* strategy yields the following distribution of points on the sphere. We see that the points are clustered around the poles ($\phi = 0$ and $\phi = \pi$) and sparse around the equator ($\phi = \pi/2$).



The reason for this is that the area dA of a differential surface element in spherical coordinates is $dA(d\theta, d\phi) = r^2 \sin(\phi)d\phi d\theta$. This formula for the area of a differential surface element comes from treating it as a square of dimension $r d\phi$ by $r \sin(\phi) d\theta$. These dimensions of the differential surface element come from simple trigonometry.



So, close to the poles of the sphere ($\phi = 0$ and $\phi = \pi$), the differential surface area element determined by $d\theta$ and $d\phi$ gets smaller since $\sin(\phi) \rightarrow 0$. Thus, we should include less points near $\phi = 0$ and $\phi = \pi$ and more points near $\phi = \pi/2$ to achieve a uniform distribution on the sphere.

Our goal is to find and then draw samples from the probability distribution $f(\theta, \phi)$ that maps from the θ - ϕ plane to a uniform distribution on the sphere.

Let v be a point on the unit sphere S . We want the probability density $f(v)$ to be constant for a uniform distribution. Thus $f(v) = \frac{1}{4\pi}$ since $\int \int_S f(v) dA = 1$ and $\int \int_S dA = 4\pi$. We want to represent points v using the parameterization in θ and ϕ and find the corresponding probability density function $f(\theta, \phi)$ that maps to a uniform distribution on the sphere. We can obtain a uniform distribution by enforcing:

$$f(v)dA = \frac{1}{4\pi}dA = f(\theta, \phi)d\theta d\phi,$$

since $f(v)dA$ is the probability of finding a point in an area dA about v on the sphere. Because $dA = \sin(\phi)d\phi d\theta$, it follows that $f(\theta, \phi) = \frac{1}{4\pi}\sin(\phi)$.

Marginalizing the joint distribution to get the p.d.f of θ and ϕ separately:

$$f(\theta) = \int_0^\pi f(\theta, \phi)d\phi = \frac{1}{2\pi}$$

$$f(\phi) = \int_0^{2\pi} f(\theta, \phi)d\theta = \frac{\sin(\phi)}{2}.$$

We see that θ is a uniformly distributed variable and $f(\phi)$ scales with $\sin(\phi)$; we want more points around the equator, $\phi = \pi/2$, which is where $\sin(\phi)$ takes its maximum.

Now, how can we sample numbers ϕ that follow the distribution $f(\phi)$? We'd like to use the readily available uniform random number generator in $[0, 1]$ as before. Inverse Transform Sampling is a method that allows us to sample a general probability distribution using a uniform random number. For this, we need the cumulative distribution function of ϕ :

$$F(\phi) = \int_0^\phi f(\hat{\phi})d\hat{\phi} = \frac{1}{2}(1 - \cos(\phi)).$$

Keep in mind that $F(\phi)$ is a monotonically increasing function from $[0, \pi] \rightarrow [0, 1]$ since it is a cumulative distribution function. Thus, it has an inverse function F^{-1} .

Let U be the uniform random number in $[0, 1]$ that we *do* know how to generate. To see how inverse transform sampling works, note that

$$Pr(U \leq F(\phi)) = F(\phi).$$

This is a property of the uniform random variable $U[0, 1]$, since for any number $x \in [0, 1]$, $Pr(U \leq x) = x$. As F is invertible and monotone, we can preserve this inequality by writing:

$$\Pr(F^{-1}(U) \leq \phi) = F(\phi).$$

Aha! This shows that $F(\phi)$ is the cumulative distribution function for the random variable $F^{-1}(U)$! Thus, $F^{-1}(U)$ follows the same distribution as ϕ . The algorithm for sampling the distribution $f(\phi)$ using inverse transform sampling is then:

- Generate a uniform random number u from the distribution $U[0, 1]$.
- Compute ϕ such that $F(\phi) = u$, i.e. $F^{-1}(u)$.
- Take this ϕ as a random number drawn from the distribution $f(\phi)$.

In our case, $F^{-1}(u) = \arccos(1 - 2u)$.

The algorithm below in C++ shows how to generate uniformly distributed numbers on the sphere using this method:

```
#include<random>
#include<cmath>
#include<chrono>

int main(int argc, char *argv[]) {
    // Set up random number generators
    unsigned seed = std::chrono::system_clock::now().time_since_epoch().count(
        std::mt19937 generator (seed);
    std::uniform_real_distribution<double> uniform01(0.0, 1.0);

    // generate N random numbers
    int N = 1000;

    // the correct way
    FILE * correct;
    correct = fopen("correct.csv", "w");
    fprintf(correct, "Theta,Phi,x,y,z\n");
    for (int i = 0; i < N; i++) {
        // incorrect way
```

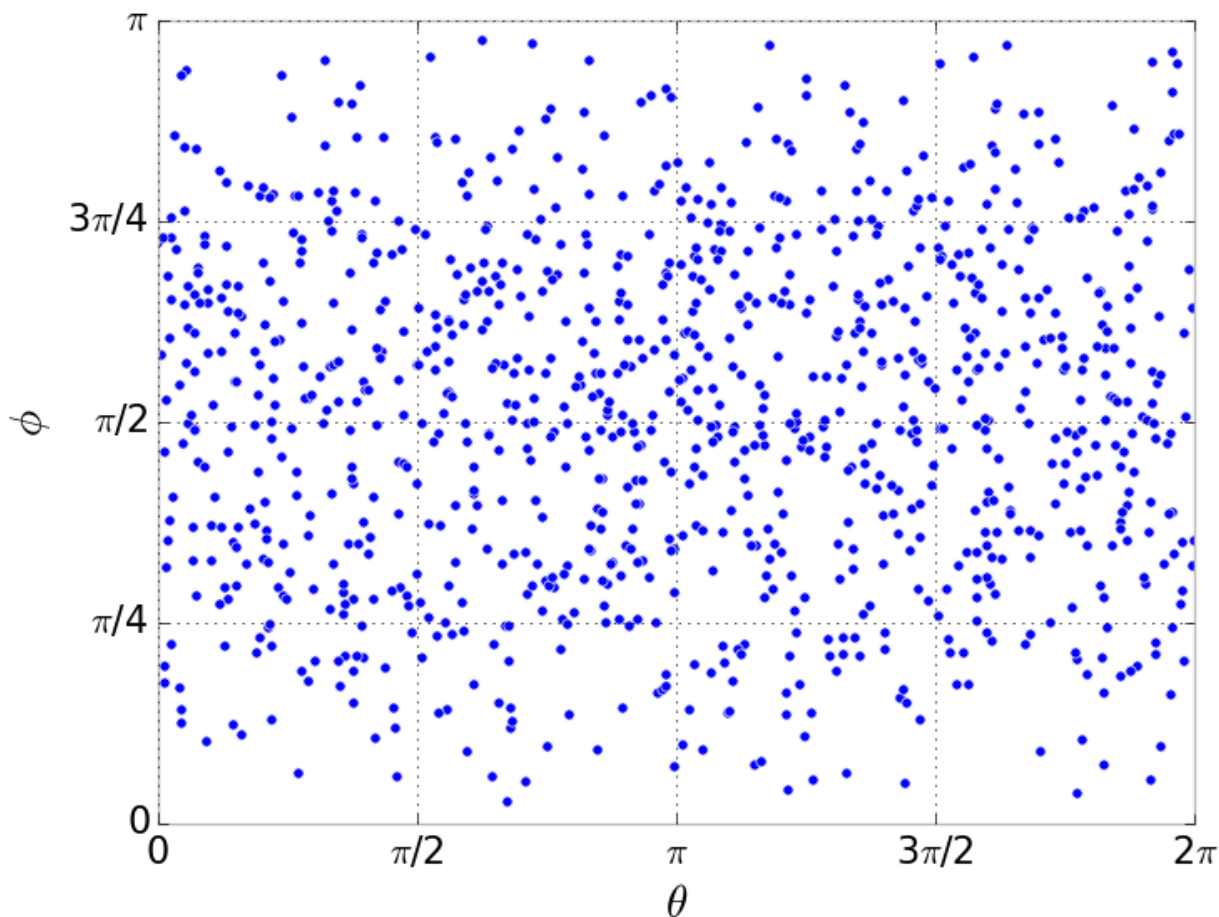


```

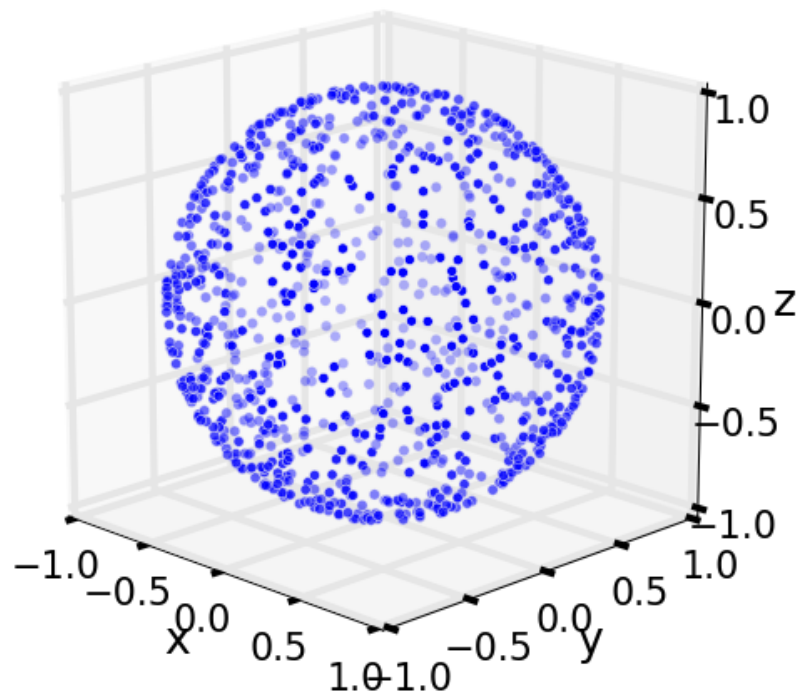
    double theta = 2 * M_PI * uniform01(generator);
    double phi = acos(1 - 2 * uniform01(generator));
    double x = sin(phi) * cos(theta);
    double y = sin(phi) * sin(theta);
    double z = cos(phi);
    fprintf(correct, "%f,%f,%f,%f,%f\n", theta, phi, x, y, z);
}
fclose(correct);
}

```

We then get the following distribution of points in the (θ, ϕ) plane. There are more points around $\phi = \pi/2$ (the equator) than around the poles ($\phi = 0, \pi$), as we had hoped for.



And, finally, a uniform distribution of points on the sphere.



Alternative method 1

An alternative method to generate uniformly distributed points on a unit sphere is to generate three standard normally distributed numbers X , Y , and Z to form a vector $V = [X, Y, Z]$. Intuitively, this vector will have a uniformly random orientation in space, but will not lie on the sphere. If we normalize the vector $V := V/\|V\|$, it will then lie on the sphere.

The following Julia code implements this. We have to be careful in the case that the vector has a norm close to zero, in which we must worry about floating point precision by dividing by a very small number. This is the reason for the `while` loop.

```
n = 100
```

```

f_normal = open("normal.csv", "w")
write(f_normal, "x,y,z\n")

for i = 1:n
    v = [0, 0, 0] # initialize so we go into the while loop

    while norm(v) < .0001
        x = randn() # random standard normal
        y = randn()
        z = randn()
        v = [x, y, z]
    end

    v = v / norm(v) # normalize to unit norm

    @printf(f_normal, "%f,%f,%f\n", v[1], v[2], v[3])
end

```

To prove this, note that the standard normal distribution is:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

As X , Y , and Z each follow the standard normal distribution and are generated independently:

$$f(x, y, z) = f(v) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \right) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \right) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \right).$$

With some algebra:

$$f(x, y, z) = f(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(x^2+y^2+z^2)} = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}\|v\|^2}.$$

This shows that the probability distribution of v only depends on its magnitude and not any direction θ and ϕ . The vectors v are thus indeed pointing in uniformly random directions. By finding where the ray determined by this vector v intersects

the sphere, we have a sample from a uniform distribution on the sphere.

Alternative method 2

Credit to [FX Coudert](#) for pointing this out in a comment, another method is to generate uniformly distributed numbers in the cube $[-1, 1]^3$ and ignore any points that are further than a unit distance r from the origin. This will ensure a uniform distribution in the region $r \leq 1$. Next, normalize each random vector to have unit norm so that the vector retains its direction but is extended to the sphere of unit radius. As each vector within the region $r \leq 1$ has a random direction, these points will be uniformly distributed on a sphere of radius 1.

Alternative method 1 can be used to *efficiently* generate uniformly distributed numbers on a hypersphere— i.e. in higher dimensions. On the other hand, as the number of dimensions grows, the ratio of the volume of the edges of a cube to the volume of the ball inside of it grows (see [Wikipedia article](#)). Hence, a larger and larger fraction of the uniformly generated numbers in the cube will be rejected using Alternative method 2, and so this algorithm will be inefficient.

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OR SIGN UP WITH DISQUS Name Sort by Best   7 **spacetunnel** • 2 years ago

Your "correct way" code has //incorrect way in the middle of it which is quite confusing

4  |  • Reply • Share ›

**cbunix23** • 4 years ago

I'd suggest using a unique random number generator and unique seed per dimension, otherwise you can get patterns in the selected points. That might not be an issue with `uniform_real_distribution` but it is with others.

2  |  • Reply • Share ›

**Karl Irikura** • a year ago

Marsaglia, G. Choosing a Point from the Surface of a Sphere. Ann. Math. Statist. 1972, 43, 645.

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**Bad Max** • 4 years ago • edited

Why Gaussians for alternative method 1? Why not uniforms on $[-1, 1]$? The direction is also random and since we normalize the vector the magnitude will also always be 1.

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**Álvaro Ridruejo**  Bad Max • 2 years ago

This would provide a wrong distribution. Since the ball is inscribed inside the cube, there are few points outside the ball near the center of the cube faces, but a great deal of them near the cube corners. By using uniform distributions, you would project all points (inside and outside the ball) onto the sphere. Instead of a uniform distribution of points, your sphere would have an artificial accumulation of points in the 8 regions closest to the cube corners. Since all points outside the ball are removed in Alternative 2, this problem is solved. Of course, Alternative 1 is also safe, because the product of the mutually orthogonal 1D normal distributions is a purely radial distribution (see proof).

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Jimmy → Tathagata Ghosh • 4 years ago

Since I've worked with [Previous](#) [Next](#) idea (which is very brute force) would be to generate N points, and apply k-means with the N number of points you need. But I'd also like to know a better way

^ | v • Reply • Share ›

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plantfx • 5 years ago

Pretty cool! The simplest way to know for a random distribution on a unit sphere is to project from a cylinder, which is an equal area projection (Lambert cylindrical equal-area projection).

So random points (u,v) represent points scattered on the surface of a cylinder

u is a random number from 0 to 2π

v is a random number from -1 to 1

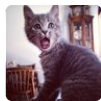
$\sin(u)X$ and $\sin(u)Z$ forms the circle that makes the cylinder and the vY gives the height. To project onto the sphere you just scale the radius of the circle, and based on $a^2+b^2=c^2$ that radius is $\sqrt{1-v^2}$.

We end up with:

$$r = \sqrt{1-v^2}$$

$$P = (r \cdot \sin(u), v, r \cdot \cos(u))$$

^ | v • Reply • Share ›



JonnyOnTheSpot • 5 years ago

Saved my life.

^ | v • Reply • Share ›



frygge • 5 years ago • edited

Do you know already whether the solution is extendable to higher dimensionalities?

^ | v • Reply • Share ›



Ernesto Mainegra • 5 years ago • edited

Great article Cory!

For a more efficient implementation, I would sample directly $\cos(\phi)$ and get

$$\sin(\phi) = \sqrt{1 - \cos(\phi)^2}.$$

Thanks!

^ | v • Reply • Share ›

1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | 125 | 126 | 127 | 128 | 129 | 130 | 131 | 132 | 133 | 134 | 135 | 136 | 137 | 138 | 139 | 140 | 141 | 142 | 143 | 144 | 145 | 146 | 147 | 148 | 149 | 150 | 151 | 152 | 153 | 154 | 155 | 156 | 157 | 158 | 159 | 160 | 161 | 162 | 163 | 164 | 165 | 166 | 167 | 168 | 169 | 170 | 171 | 172 | 173 | 174 | 175 | 176 | 177 | 178 | 179 | 180 | 181 | 182 | 183 | 184 | 185 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Wito Engelke → Cory Simon • 5 years ago

Stuped me! Thanks for pointing it out! I need it only for R^3 so I will reject something around the half of it, which is fine.

^ | v • Reply • Share ›



Cory Simon Mod → Wito Engelke • 5 years ago

Precisely:

volume of cube $[-1,1]^3 = 8$

volume of sphere, radius 1: $\frac{4}{3} * \pi$

So the uniform point in the cube will be in the sphere with probably $\frac{4}{3} * \pi / 8 \approx 0.52!$

^ | v • Reply • Share ›



Faisal Khan → Cory Simon • 2 years ago

How can I cite this work in my research work ????

^ | v • Reply • Share ›



Wito Engelke → Cory Simon • 5 years ago

Hahaha. I calculated exactly that and rounded it to a half, when writing the answer.

^ | v • Reply • Share ›



Jamileh Yousefi • 6 years ago

Hello Cory,

Thank you for sharing this nice article.

How can I cite your article .

Thanks,

Jamileh

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FX Coudert • 8 years ago



Markus Klyver • 5 years ago

Nice!