$k(k-1)$ and using the fact that $b_{12} = 45$, we obtain

$$\frac{1}{2} + \sum_{j=3}^{k} \binom{k-2}{j-2} \frac{45^{j-2}}{j(j-1)} b_{j-1} = 0. \tag{14}$$

In this case, each fraction in the sum, when reduced, has numerator divisible by 5, so that $\frac{1}{2}$ should also have this property. This provides a contradiction to the assumption that $k \geqslant 3$.

It turns out that the role of unique factorization in $\mathbb{Q}(\sqrt{-7})$ is not as critical as the exposition here might indicate. Using the unique decomposition of ideals into products of prime ideals in quadratic fields, one can apply the techniques above generally to solve, for example, all Diophantine equations of the form $X^2 + (4q - 1) = 4q^n$, where q is an arbitrary prime. For more on this, see the more-detailed paper of the author [**4**].

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THE TEACHING OF MATHEMATICS

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The Tumbling Box

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Introduction. Toss a rigid body, such as a book or an empty cereal box, in the air three times, each time giving it a spin about one of its axes. It is perhaps surprising to learn that the box will always rotate stably about two of its three axes, but will

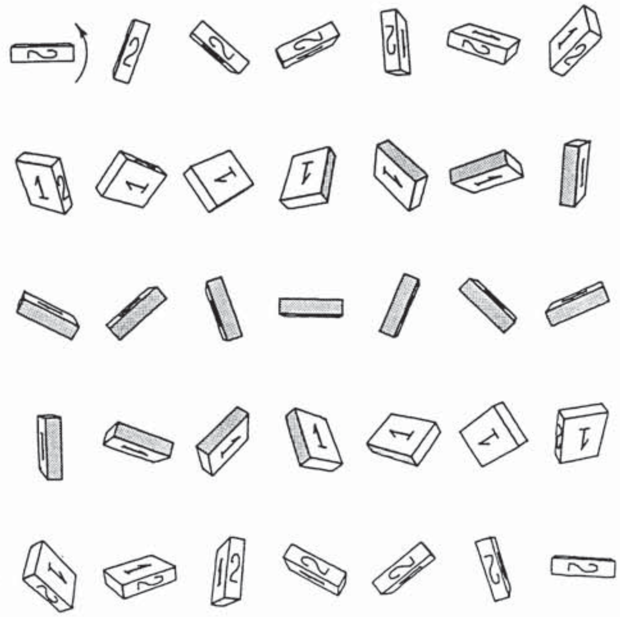

FIG. 1. Numerical plot of a box tumbling in air. The box is viewed from a distance of 5 box lengths and along the direction of the fixed angular momentum vector $\vec{L}$. In these "snapshots," the unstable axis #2 is misaligned from $\vec{L}$ by 5°. The principal moments of inertia of the box are in the ratio $I_1 : I_2 : I_3 :: 6 : 4 : 3$.

wobble and somersault unstably about the third (see Fig. 1). This fact is a well-known result from classical mechanics (see [**4**] pp. 116ff. or [**1**] pp. 142–145), but, unfortunately, one with which too few mathematicians seem to be familiar. The tumbling box problem presents a very beautiful and natural example of a system of nonlinear differential equations, one which can be made appropriate for an undergraduate course in the subject, but one which fails to appear in any of the elementary texts. In addition, it is one of those rare problems whose purely mathematical solution is also easily verified empirically.

I would like to express my gratitude to Professor Alar Toomre of M.I.T. for introducing me and countless others to this particular version of the problem (see, for example, [**2**] problem 4.51, pp. 202ff.), for encouraging me to present it here, and for allowing me to reproduce his splendid diagrams in Figs. 1 and 2.

Some Physics. The basic law of motion for any vector function of a rigid body in space is Euler's equation. If $\vec{A}(t)$ is any vector function pertaining to the body, the formula states

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{space}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{A}, \tag{1}$$

where $(d\vec{A}/dt)_{\text{space}}$, $(d\vec{A}/dt)_{\text{body}}$ are the rates of change respectively measured in fixed spatial coordinates and in coordinates relative to the principal axes of the object, and where $\vec{\omega}$ is angular velocity (see, for instance, [**3**] p. 133). This formula says essentially that $d\vec{A}/dt$ consists of both a translational and a rotational component, the latter being given by $\vec{\omega} \times \vec{A}$.

For our tumbling box, we are concerned with the case $\vec{A} = \vec{L}$, the angular momentum. If we toss the box by giving it a spin as described in the introduction, we are introducing a constant angular momentum vector. Hence $(d\vec{L}/dt)_{\text{space}} = \vec{0}$ and so (1) becomes

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{body}} = \vec{L} \times \vec{\omega}. \tag{2}$$

The Differential System. In the sequel, we assume that all coordinates are measured relative to the axes of the box and henceforth we will drop the subscript "body." Then $\vec{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$, where ω_j is the jth component of angular velocity and I_j the moment of inertia about the jth principal axis of the box. If we assume that the box is uniform and has distinct dimensions (so that the box may indeed be considered to look like a book), then we may take $I_1 > I_2 > I_3 > 0$. Then, in coordinates, (2) becomes the system

$$\begin{cases} \dot{L}_1 = (I_2 - I_3)\,\omega_2\omega_3, \\ \dot{L}_2 = (I_3 - I_1)\,\omega_1\omega_3, \\ \dot{L}_3 = (I_1 - I_2)\,\omega_1\omega_2. \end{cases}$$

Equivalently, since $\omega_j = L_j/I_j$, $j = 1, 2, 3$, we have

$$\begin{cases} \dot{L}_1 = \left(\dfrac{1}{I_3} - \dfrac{1}{I_2}\right) L_2 L_3, \\ \dot{L}_2 = \left(\dfrac{1}{I_1} - \dfrac{1}{I_3}\right) L_1 L_3, \\ \dot{L}_3 = \left(\dfrac{1}{I_2} - \dfrac{1}{I_1}\right) L_1 L_2. \end{cases} \tag{3}$$

It is easy to check that from (3) it follows that

$$L_1\dot{L}_1 + L_2\dot{L}_2 + L_3\dot{L}_3 = 0. \tag{4}$$

Integrating, we find

$$L_1^2 + L_2^2 + L_3^2 = C.$$

Hence we see that the trajectories of (3) must all lie on spheres centered at the origin. For simplicity, let us consider only the case $C = 1$. We have reduced our problem to that of finding solutions on a single *phase sphere*.

Linearizations. Now we make a standard local analysis. It is not difficult to see that (3) has six isolated critical points at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. By using Taylor's formula for several variables near each critical point, we may approximate (3) locally by the following six linear systems:

$$\text{near } (\pm 1, 0, 0) \qquad \begin{cases} \dot{L}_1 = 0, \\ \dot{L}_2 = \pm\left(\dfrac{1}{I_1} - \dfrac{1}{I_3}\right) L_3, \\ \dot{L}_3 = \pm\left(\dfrac{1}{I_2} - \dfrac{1}{I_1}\right) L_2, \end{cases} \tag{5a}$$

$$\text{near } (0, \pm 1, 0) \qquad \begin{cases} \dot{L}_1 = \pm\left(\dfrac{1}{I_3} - \dfrac{1}{I_2}\right) L_3, \\ \dot{L}_2 = 0, \\ \dot{L}_3 = \pm\left(\dfrac{1}{I_2} - \dfrac{1}{I_1}\right) L_1, \end{cases} \tag{5b}$$

$$\text{near } (0, 0, \pm 1) \qquad \begin{cases} \dot{L}_1 = \pm\left(\dfrac{1}{I_3} - \dfrac{1}{I_2}\right) L_2, \\ \dot{L}_2 = \pm\left(\dfrac{1}{I_1} - \dfrac{1}{I_3}\right) L_1, \\ \dot{L}_3 = 0. \end{cases} \tag{5c}$$

Since in each of (5a), (5b), and (5c) one of L_1, L_2, or L_3 is constant to first order, we may regard each of these linear systems as being two-dimensional by "ignoring" the constant variable. With such a simplification, the characteristic equation of (5a) is $x^2 - \alpha = 0$, where

$$\alpha = (1/I_1 - 1/I_3)(1/I_2 - 1/I_1) < 0 \quad (\text{for } I_1 > I_2 > I_3).$$

Hence the corresponding eigenvalues are pure imaginary and thus the linear critical points at $(\pm 1, 0, 0)$ are *centers*. Similarly, the characteristic equation of (5c) is $x^2 - \beta = 0$, where

$$\beta = (1/I_1 - 1/I_3)(1/I_3 - 1/I_2) < 0,$$

so at $(0, 0, \pm 1)$ we also have centers. However, (5b) has characteristic equation $x^2 - \gamma = 0$, where

$$\gamma = (1/I_2 - 1/I_1)(1/I_3 - 1/I_2) > 0.$$

Thus the eigenvalues of (5b) are real and of opposite sign and hence the critical points at $(0, \pm 1, 0)$ are *saddle points* (and are unstable).

Unfortunately, the local analysis above does not yet afford a complete solution because of the centers resulting from (5a) and (5c). A center singularity is a "borderline case" in that the original nonlinear system possibly could have a singularity of a different type (see, for example, [**2**, p. 183]). However, analogously to (4), we may check that from (3),

$$\frac{I_2 I_3}{I_2 - I_3} L_1 \dot{L}_1 + \frac{I_1 I_3}{I_1 - I_3} L_2 \dot{L}_2 = 0, \tag{6a}$$

$$\frac{I_1 I_3}{I_1 - I_3} L_2 \dot{L}_2 + \frac{I_1 I_2}{I_1 - I_2} L_3 \dot{L}_3 = 0. \tag{6b}$$

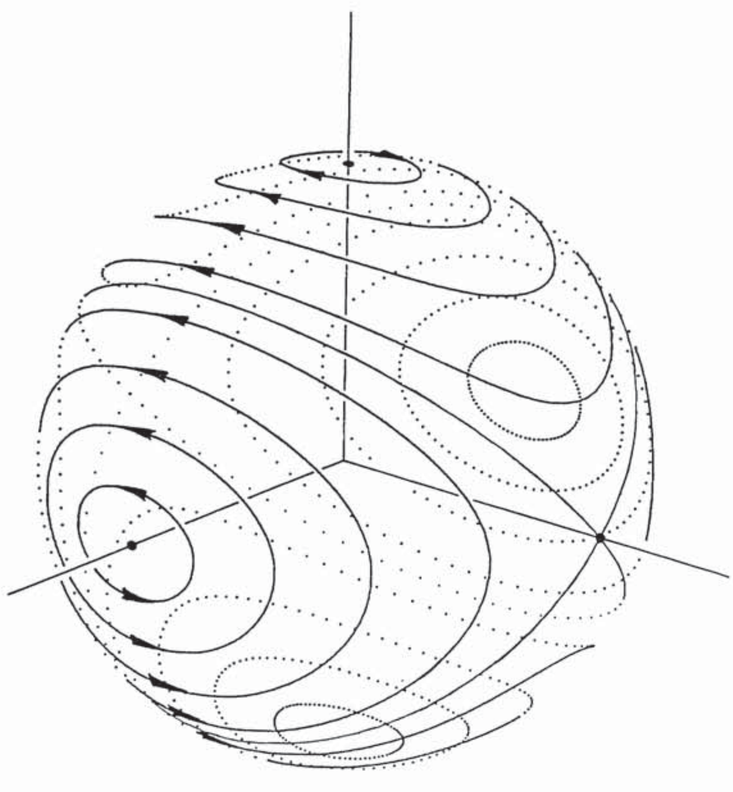

FIG. 2. Trajectories of $\left(\frac{d\vec{L}}{dt}\right)_{\text{body}} = \vec{L} \times \vec{\omega}$ on the phase sphere $L_1^2 + L_2^2 + L_3^2 = 1$.

These equations integrate respectively to

$$\frac{I_2}{I_2 - I_3} L_1^2 + \frac{I_1}{I_1 - I_3} L_2^2 = C_1, \tag{7a}$$

$$\frac{I_3}{I_1 - I_3} L_2^2 + \frac{I_2}{I_1 - I_2} L_3^2 = C_2. \tag{7b}$$

Equations (7a) and (7b) describe elliptical cylinders with axes the z and x axes, respectively. For sufficiently small C_1 and C_2, the intersections of these cylinders with the phase sphere are closed curves about the z and x axes. Hence the centers remain stable centers when we pass from the linearizations to the nonlinear system (3). See Fig. 2 for a sketch of the trajectories on the phase sphere, along with arrows indicating direction with increasing time. From this diagram, it is apparent that rotations about either the longest or the shortest axis are non-asymptotically stable, that rotations about the middle axis are unstable, and, furthermore, that this unstable motion will tend to wobble around one of the two types of stable rotations.

Finally, we remark that if any two of the principal moments of inertia are equal, then (3) immediately reduces to a simpler linear system which is easily seen to yield only stable rotations.

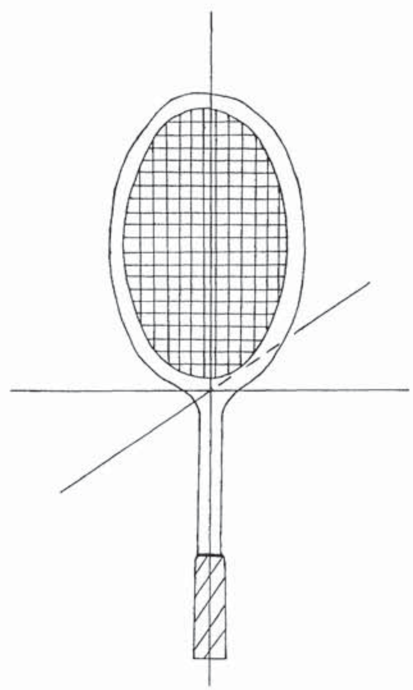

FIG. 3. The tumbling tennis racquet. Rotations about axis 2 will exhibit Eulerian wobble.

Note for Tennis Players. Essentially the same analysis as that employed above can also be applied to the tossing of tennis racquets to verify the existence of stable and unstable rotations (see Fig. 3). One finds that spinning the racquet about the axis which is perpendicular to the neck and lies in the "plane" of the racquet results in the Eulerian wobble described above. Tennis racquets provide excellent examples of rigid bodies with distinct principal moments of inertia, since they have handles which make them convenient to throw.

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The Multiplication Theorem for Fredholm Operators

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A bounded linear operator T from one Banach space to another is called a Fredholm operator if its kernel is of finite dimension and its range is of finite codimension; one then defines $\mathrm{ind}(T)$, the index of T, to be the difference between the dimension of the kernel of T and the codimension of the range of T. A Fredholm operator automatically has a closed range, a property that in many treatments is incorporated as part of the definition. The resulting redundancy is not a practical disadvantage, for, invariably, in checking in a concrete case that an operator satisfies the requirements of the definition, when the range of the operator is not obviously closed, one shows it is closed as part of the argument that shows it has a finite codimension. Moreover, the instructor who uses the longer definition can, without interrupting the logical flow of the lectures, assign to the students the task of proving that the two definitions are equivalent. The proof is a satisfying application of the open mapping theorem **[8]**.

The theorem referred to in the title is one of the central results about Fredholm operators. It states: *If T is a Fredholm operator from X to Y and S is a Fredholm operator from Y to Z, then ST is a Fredholm operator whose index is the sum of the indices of S and T.* The main point I wish to make here is that this theorem is a purely algebraic result whose general case is easily reduced to the finite-dimensional one, that case being an immediate consequence of the fundamental theorem of linear algebra (which states: *For a linear operator acting on a finite-dimensional vector space, the dimensions of the kernel and the range add up to the dimension of the space*). I was prompted to write this note because the proofs of the multiplication