The telegraph equation[1]

The equation in terms of voltage $V = V(t,x)$:

$$V_{xx} = LC_0V_{tt} + \left(R_cC_0 + \frac{L}{R_0}\right)V_t + \frac{R_c}{R_0}V \tag{1}$$

where

- L is the self-inductance of the cable,
- C_0 is the capacitance of the insulator,
- R_0 is the resistance of the insulator,
- R_c is the resistance of the cable.

All are assumed to be constant throughout the length of the cable [that is, the cable is assumed to be homogenous]. It so happens that L, C_0, and R_c are measured in the corresponding units [Henry, Farad, Ohm] per unit length, whereas R_0 is measured in *Ohm* · *meter*. This becomes clear during the derivation or simply by trying to reconcile the units in equation (1).

Derivation: an outline. Beside voltage $V = V(t,x)$ and current $I = I(t,x)$ in the cable at time t and point x, there is also the *leakage* current $j = j(t,x)$, per unit length of the cable so that

$$I(t, x + \triangle x) - I(t,x) \approx j(t,x)\triangle x,$$

that is,

$$j(t,x) = -I_x(t,x). \tag{2}$$

On the other hand, leakage is caused by the finite resistance and non-zero capacitance of the insulator of the cable:

$$j(t,x)\triangle x \approx \left(\frac{V(t,x)}{R_0} + C_0V_t(t,x)\right)\triangle x,$$

that is,

$$j(t,x) = \frac{V(t,x)}{R_0} + C_0V_t(t,x). \tag{3}$$

Finally, we have the basic equation for the voltage drop according to the laws of Ohm and Faraday:

$$V(t,x) - V(t, x+\triangle x) \approx \Big(R_cI(t,x) + LI_t(t,x)\Big)\triangle x,$$

that is,

$$-V_x(t,x) = R_cI(t,x) + LI_t(t,x). \tag{4}$$

Now, to get (1), we eliminate I and j: differentiate both sides of (4) with respect to x, substitute j for $-I_x$ [from (2)] and then use (3) to write everything in terms of V.

As a quick concept check, confirm that, by eliminating j and V, you get an equivalent form of (1) in terms of the current I:

$$I_{xx} = C_0LI_{tt} + \left(\frac{L}{R_0} + R_cC_0\right)I_t + \frac{R_c}{R_0}I.$$

Equation (1) is considered on the half-line $x > 0$ with zero initial conditions $V(0,x) = V_t(0,x) = 0$, and the boundary condition $V(t,0) = f(t)$ represents the signal being transmitted.

The *ideal* cable has not active leakage and no active resistance: $R_0 = +\infty$, $R_c = 0$, which turns (1) into a wave equation

$$V_{tt} = c^2V_{xx},\ t > 0,\ x > 0,\ V|_{x=0} = f(t), \tag{5}$$

[1] Sergey Lototsky, USC

where $c = \frac{1}{\sqrt{LC_0}}$. It is not the standard wave equation on the line [it is on the half line, with a boundary condition], but it can be reduced to one using the *method of reflection*. The final answer can be verified directly:

$$V(t,x) = \begin{cases} f\left(t - \frac{x}{c}\right), & x < ct \\ 0 & x \geq ct. \end{cases} \tag{6}$$

In other words, the initial signal propagates to the right along the cable with speed c, so that you are receiving exactly what was transmitted.

The wave equation (5) is an acceptable model for telegraph [above-the-ground] wires with amplifiers along the way, as well as reasonably short underwater cables [was good enough for the 100km long Great Britain-to-France cable]. For longer underwater cables, one can still ignore the active leakage [$R_0 = +\infty$] but can no longer ignore the active resistance of the cable. In fact, active resistance becomes dominant in the sense that LC_0V_{tt} becomes much smaller than $R_cC_0V_t$. As a result, equation (5) becomes the *heat* equation

$$V_t = aV_{xx},\ t > 0,\ x > 0,\ V|_{x=0} = f(t), \tag{7}$$

where $a = \frac{1}{R_cC_0}$. The closed-form solution can be written using the method of reflection, but it is not as nice as (6):

$$V(t,x) = \int_0^t \frac{x}{\sqrt{4\pi a(t-s)^3}}\, e^{-x^2/(4a(t-s))} f(s)\, ds.$$

This equality is highly non-trivial: note that you cannot even set $x = 0$ on the right-hand side; instead, a rather sophisticated computation shows that, if $x \to 0+$, then, under some additional assumptions about the function f, the right-hand side approaches $f(t)$.

The bottom line: in a long underwater cable, the signal becomes "blurred" and a "point signal" transmitted at the source $x = 0$ [think one dot or one dash of the Morse code] will be "going on" for about x_d^2/a seconds when received at (large) distance x_d from the source. Thus, the key to a good underwater cable is large a, or, equivalently, small R_c. It was Sir William Thomson, 1st Baron Kelvin, who figured it all out, thus ensuring the resounding success of the [second] transatlantic cable, getting rich, and [potentially] changing the course of the world history.