STOCHASTIC PERRON'S METHOD FOR HAMILTON-JACOBI-BELLMAN EQUATIONS

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Abstract. We show that the value function of a stochastic control problem is the unique solution of the associated Hamilton-Jacobi-Bellman (HJB) equation, completely avoiding the proof of the so-called dynamic programming principle (DPP). Using Stochastic Perron's method we construct a super-solution lying below the value function and a sub-solution dominating it. A comparison argument easily closes the proof. The program has the precise meaning of verification for viscosity-solutions, obtaining the DPP as a conclusion. It also immediately follows that the weak and strong formulations of the stochastic control problem have the same value. Using this method we also capture the possible face-lifting phenomenon in a straightforward manner.

1. Introduction

Stochastic Perron's method was introduced in [2] for linear problems, and adapted in [1] to free-boundary problems associated to Dynkin games. In the present paper, we carry out a similar line of ideas, but with significantly different technicalities, to the most important case of non-linear problems, namely that of Hamilton-Jacobi-Bellman equations (HJB) in stochastic control. The result presented here actually represents the original motivation to introduce the stochastic version of Perron's method.

The goal of the paper is

1. to prove the general result stating that "the value function of a control problem is the unique viscosity of the associated HJB",
2. *without* having to first go through the proof of the Dynamic Programming Principle (DPP) but actually obtaining it as a by-product,
3. in an as much *elementary* manner as possible.

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The motivation for such goal is described in detail in [2] and [1]. To summarize, the program described by (1) and (2) (and, implicitly (3)) amounts to a verification result for viscosity solutions of HJB's. Overall, we believe this to be a genuinely novel approach to stochastic control, that provides a deeper understanding of the relation between controlled diffusions and (viscosity) solutions of HJB's.

In addition to being a new method to treat a fundamental class of problems, we believe the program carried out here has two notable features, which basically amount to achieving our goals above:

1. This is a direct/verification approach to dynamic programming (similar to [19] or [20]), in that it *first* finds/constructs a solution to the HJB, *then* shows that such solution is the value function, avoiding altogether the Dynamic Programming Principle. However, this is technically very different from the verification approach in [19] or [20], and can be viewed as a probabilistic counterpart to the classical approach.
2. We believe it, indeed, to be more elementary than either going through the probabilistic proof of the DPP (which is often incomplete, as described in the recent paper [5] where some important details are fixed) and then having to prove comparison of viscosity solutions anyway or through the analytical techniques in [19] and [20]. In particular, there is no need for us to use "conditional controls" or canonical spaces, usually needed in the proof of the DPP. These arguments are still needed even in the recent proof of Bouchard and Touzi of a weaker version of the DPP [3]. While measurable selection arguments are circumvented there through the weaker formulation, the Markov arguments mentioned above are still present. Our method consists only in applying Itô's formula along the smooth test functions for viscosity solutions, plus an elementary stopping argument. In addition, arguments of the same type as we use here (maybe even more complicated) have to be used *anyway* when one uses the weak DPP to prove that the value function is a viscosity solution. Also, we avoid the technicalities related to approximation by convolution and the approximation of the state equation in [19] and [20].

We present here a fresh look at a classic problem, so some comments on the existing literature are needed. We mention briefly only those works that are closest related, at the risk of omitting relevant but further ideas. We first start with some important work in stochastic control, which, in the same spirit as our paper, avoids the DPP.

Since our result amounts to verification without smoothness, it is conceptually closest to [19] and [20]. Using approximation by convolution of viscosity semi-solutions in the deterministic case ([19]) and then also approximating the state equation by non-degenerate diffusions in the stochastic framework of two-player games ([20]), the author performs a verification argument arriving at similar conclusions (in different situations though). The probability space needs to accommodate an additional Brownian motion in the stochastic case, and, as mentioned above, the technicalities are very different and more involved, compared to our approach. Overall, the two approaches have little, if anything, in common.

At a formal level, one of our main results, Theorem 4.1, looks very much like the main result in the seminal work of Fleming-Vermes [13] and [12] (see also Remark

5.5). [1] More precisely, while the authors in [13] and [12] show that the value function is the infimum of classical super-solutions, we show that, the value function is below the infimum of stochastic super-solutions, which is a viscosity sub-solution. While appearing stronger than our Theorem 4.1 (considered by itself, without the other main result Theorem 3.1), the notable result in Fleming-Vermes has two features:

1. It contains a sophisticated approximation/separation argument used on top of re-stating the optimization problem as an infinite dimensional convex program,
2. It still uses the very definition of the value function,
3. By itself, is not enough to show the value function is a viscosity sub-solution. Even if one does not mind the complicated approximation arguments, our Theorem 4.1 is still needed on top of the very strong results in [13] and [12] to get such a conclusion. Even combining Fleming-Vermes with the Perron's method in Ishii [14] would not yield this: the infimum over viscosity super-solutions may go below the value function, unless we now a *a-priori* that the value function is a viscosity sub-solution, and we also have a comparison result (needed even for the viscosity version of Perron in [14]). A sub-approximation counterpart to the work of Fleming-Vermes could close the argument, but this would still have a very different flavor than our work, since it uses, once again, the representation of the value function. Actually, the recent papers [9, 10, 11] carry along these lines, for path-dependent HJB's.

If one attempts to only use the Perron's method in Ishii [14] to construct viscosity solutions, the same obvious obstacle described in relation to Fleming-Vermes arises: without additional knowledge on the properties of value function, it does not compare with the output of Perron's method.

It should be also mentioned how our result compares to other existing results about verification for viscosity solutions of HJB's, namely [21]. The result in [21] starts from the fact that the value function is the unique viscosity solution, and, using this piece of information, synthesizes the optimal control (if one exists) in terms of the generalized derivatives of the viscosity solution/value function. Our result plays a role *before* the synthesis described in [21], and proves exactly that the value function is the unique viscosity solution, *without* resorting to the use of DPP. In other words, our work addresses a different question than the one addressed in [21] (but quite similar to [19] and [20]).

The rest of the paper is organized as follows: In Section 2, we present the basic setup of the stochastic control problem, introduce the related HJB and the terminal condition. Moreover, we state our assumptions on the Hamiltonian. In Section 3, we consider the strong formulation of the stochastic control problem and introduce the class of stochastic sub-solutions via which we construct a lower bound on the value function which is a viscosity super-solution. In Section 4, we introduce the weak formulation of the stochastic control problem and introduce the class of stochastic super-solutions using which we construct a viscosity sub-solution to the HJB equation. Finally, in Section 5, we verify that both value functions, in the weak and the strong formulation, equal the unique viscosity solution using comparison.

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2. Setup

Let U be a closed subset of $\mathbb{R}^k$ (the control space) and $\mathcal{O}$ an open subset of $\mathbb{R}^d$ (the state space). Let $b : [0,T] \times \mathcal{O} \times U \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathcal{O} \times U \to \mathbb{M}_{d,d'}$ be two measurable functions. We consider the controlled diffusion

$$dX_t = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t, \quad X \in \mathcal{O}. \tag{2.1}$$

We assume that the state lives in the open domain $\mathcal{O} \subset \mathbb{R}^d$, to include the treatment of utility maximization models for positive wealth, which is popular in mathematical finance. Given a measurable function $g : \mathcal{O} \to \mathbb{R}$, our goal is to maximize the expected payoff received at a fixed time-horizon $T > 0$ using progressively measurable processes u taking values in U. Informally, we want to study the optimization problem

$$\sup_u \mathbb{E}[g(X_T^u)], \quad X_0 = x \in \mathcal{O}.$$

Remark 2.1. *We choose only to maximize terminal payoffs, just to keep the notation simpler. In the literature, this is known as the Mayer formulation of stochastic control problems. The Bolza problem, which contains a running payoff as well, can be treated in an identical manner, with some additional notation.*

One associates the following Hamiltonian to this problem:

$$H(t,x,p,M) := \sup_{u \in U} \left[b(x,u) \cdot p + \frac{1}{2} Tr(\sigma(x,u)\sigma(x,u)^T M) \right], \quad 0 \leq t \leq T, \ x \in \mathcal{O}.$$

We make the following assumption on the Hamiltonian:

Assumption 2.1. *Let us denote the domain of H by*

$$dom(H) := \{(t,x,p,M) \in [0,T] \times \mathcal{O} \times \mathbb{R}^d \times \mathcal{S}_d : H(t,x,p,M) < \infty\}.$$

We will assume that H is continuous in the interior of $dom(H)$. Moreover, we will assume that there exists a continuous function $G : [0,T] \times \mathcal{O} \times \mathbb{R}^d \times \mathcal{M}_d \to \mathbb{R}$ such that

1. $H(t,x,p,M) < \infty \implies G(t,x,p,M) \geq 0$,
2. $G(t,x,p,M) > 0 \implies H(t,x,p,M) < \infty$.

Remark 2.2. *Our assumption above on the Hamiltonian H differs from that of* [16]*, which assumes that the domain of H is closed. This latter assumption is well-suited for analyzing super-replication problems with volatility uncertainty but excludes the utility maximization problems. For example, our assumption works out well for utility maximization problems, where $\mathcal{O} = (0,\infty)$ and $G(t,x,p,M) = -M$. Of course, one may ask why not simply choose $G = e^{-H}$? This is because, in general, H is not jointly continuous everywhere as an extended value function. For example, in the case of one-dimensional utility maximization, where $H(t,x,p,M) = -p/2M^2$ for $M < 0$, one can see that the Hamiltonian is not continuous at $(p,M) = (0,0)$, even if we view it as extended-valued. If H is continuous everywhere, as an extended-valued mapping, then we can, indeed, choose $G = e^{-H}$. However, this is usually not the case.*

Using the Stochastic Perron's Method, our goal is to show that, when a comparison principle is satisfied, the value function is, immediately, the unique viscosity solution of

$$\min\{-v_t(t,x) - H(t,x,v_x(t,x),v_{xx}(t,x)), G(t,x,v_x(t,x),v_{xx}(t,x))\} = 0, \tag{2.2}$$

for $(t,x) \in [0,T) \times \mathcal{O}$, with the terminal condition

$$\min[v(T,x) - g(x), G(T,x,v_x(T,x),v_{xx}(T,x))] = 0, \quad \text{on} \quad \mathcal{O}, \tag{2.3}$$

without having to prove the dynamic programming principle.

Remark 2.3. *One may question why we do not impose any kind of boundary conditions on $\partial\mathcal{O}$. This is because, as we can see from the assumptions below, we choose $\mathcal{O}$ as a natural domain, so that the controlled state process X never makes it to the boundary.*

3. Stochastic sub-solutions

In this section we will consider the so-called "strong formulation" of the stochastic control problem.

The main goal of the paper is to outline how the Stochastic Perron's Method in [2] and [1] can be used for the more important problem of Hamilton-Jacobi-Bellman equations. Having such goal in mind, but wanting to keep the presentation simpler, we make *quite restrictive assumptions, without losing the very interesting case when a boundary layer is present.* However, the restrictive assumptions we make are actually present in the important examples we have in mind. Our analysis can be carried through under weaker assumptions, but, as it is customary in stochastic control, this would have to be done on a case by case basis, adapting the method to the specific optimization problem. This is particularly important as far as admissibility is concerned.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting an $\mathbb{R}^{d'}$-valued Brownian motion. Given T let $\mathbb{F} := \{\mathcal{F}_t,\ 0 \leq t \leq T\}$ denote the completion of the natural filtration of this Brownian motion. (Note that $\mathbb{F}$ satisfies the usual conditions.)

Assumption 3.1 (State Equation). *For any $(t,u) \in [0,T] \times U$ and $x, y \in \mathbb{R}^d$ we have*

$$\begin{aligned} |b(t,0,u)| + |\sigma(t,0,u)| &\leq C(1+|u|), \\ |b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| &\leq L(|u|)|x-y|, \end{aligned} \tag{3.1}$$

for some constant C and some non-decreasing function $L : [0,\infty) \to [0,\infty)$.

In what follows, we will work with controls and solutions defined on stochastic intervals. It is well know that, for deterministic intervals, one can choose integrands which are progressively measurable, optional or predictable, as they are equal up to equivalent classes. We choose here to work predictable controls, which are both the most general (i.e. work even for jump-diffusions) and best suited to handle joint-measurability in (t,ω) that is required on stochastic intervals.

Admissibility (i.e. bounds or integrability) is another very important issue, and we choose here a very small class of admissible process, namely bounded controls, but the bound is not fixed a-priori (unless the control space U is itself compact). This allows to capture the full behavior of the value function, i.e. face lifting phenomenon, but the optimal control may not be admissible, if such a control exists. This choice of admissible controls is the same as in Section 6 of Krylov [15], for the case of unbounded controls.

Definition 3.1. *Let $0 \leq \tau \leq \rho \leq T$ be stopping times. By $\mathcal{U}_{\tau,\rho}$ we denote the collection of predictable processes $u : (\tau, \rho] \to U$, by which we mean that the joint* *map*

$$(0,T] \times \Omega \ni (t,\omega) \to u_t(\omega) \times 1_{[\tau(\omega)<t\leq\rho(\omega))}$$

is predictable with respect to the filtration $\mathbb{F}$ *and which are uniformly bounded, i.e. there exists a positive constant* $0 \leq B(u) < \infty$ *such that*

$$\|u\| := \sup_{\tau(\omega)\leq t<\rho(\omega)} |u_t(\omega)| \leq B(u).$$

Our definition of admissible control is very restrictive, in order to be able to deal simultaneously with a reasonably large class of problems. Of course, with this definition one does not expect an admissible optimal control to exist. However, if particular problems are considered, the definition of admissibility can be changed to a larger class that does contain the optimal control (if such exists). For example,

(1) in the case of utility maximization, controls should only be locally integrable, and admissibility is a state constraint, namely that the wealth is non-negative,
(2) in the case of classical quadratic-type energy minimization, controls should be square integrable.

Our proofs work verbatim in these particular cases.

Remark 3.1. *Assumption 3.1 on the controlled SDE, together with the Definition 3.1 ensures that there is always a unique strong (adapted to* $\mathcal{F}_t$*) solution to the controlled SDE up to an explosion time. The additional Assumption 3.2 below actually means that there is never an explosion (for bounded controls). This is always the case if* $\mathcal{O} = \mathbb{R}^d$*, or in the case of utility maximization, if the control is the proportion of stocks held.*

Assumption 3.2 (**Natural Domain**)**.** *For any stopping times* $\tau \leq \rho$ *and any initial condition* $\xi \in \mathcal{F}_\tau$ *satisfying* $\mathbb{P}(\xi \in \mathcal{O}) = 1$*, if* $u \in \mathcal{U}_{\tau,\rho}$*, the unique strong solution* $X^{u;\tau,\xi}$ *of the SDE*

$$\left\{ \begin{array}{l} dX_t = b(t,X_t,u_t)dt + \sigma(t,X_t,u_t)dW_t, \ \ \tau \leq t \leq \rho, \\ X_\tau = \xi \end{array} \right. \tag{3.2}$$

does not explode, i.e. $\mathbb{P}(X_t^{u;\tau,\xi} \in \mathcal{O}, \quad \tau \leq t \leq \rho) = 1$.

We denote $\mathcal{U}_{0,T}$ by $\mathcal{U}$. Then let us define the value function by

$$V(t,x) = \sup_{u\in\mathcal{U}_{t,T}} \mathbb{E}[g(X_T^{u;t,x})], \quad 0 \leq t < T, \ x \in \mathcal{O}.$$

The goal of this section is to construct a super-solution of the Hamilton-Jacobi-Bellman equation (2.2) with the terminal condition (2.3) that is smaller than the value function V. In order to do that, we need some growth property to be imposed on the pay-off function g and the potential solutions of the PDE. In this direction, we make an additional assumption:

Assumption 3.3 (**Growth in** x)**.** *There exists a continuous and strictly positive gauge function* $\psi : \mathcal{O} \to (0,\infty)$ *such that*

(1) *for any* $\tau \leq \rho$ *and any initial condition* $\xi \in \mathcal{F}_\tau$, $\mathbb{P}(\xi \in \mathcal{O}) = 1$*, which satisfies* $\mathbb{E}[\psi(\xi)] < \infty$*, if the control u is admissible, i.e.* $u \in \mathcal{U}_{\tau,\rho}$*, then*

$$\mathbb{E}\left[\sup_{\tau\leq t\leq\rho} \psi(X_t^u)\right] < \infty\,;$$

(2) $|g(x)| \leq C\psi(x)$ *for some* C.

The assumption above is tailor-made to deal simultaneously with quadratic problems ($\mathcal{O} = \mathbb{R}^d, \psi(x) = |x|^2$ **or** $\psi(x) = 1 + |x|^2$) and utility maximization ($\mathcal{O} = (0, \infty)$, $\psi(x) = x^p$ or $\psi(x) = 1 + x^p$, $-\infty < p < 1$, $p \neq 0$). However, the choice of ψ does matter, especially in the comparison principle that we need for the terminal condition (see Remark 5.1).

Definition 3.2. *The set of stochastic sub-solutions for the parabolic PDE* (2.2)*, denoted by* $\mathcal{V}^-$*, is the set of functions* $v : [0,T] \times \mathcal{O} \to \mathbb{R}$ *which have the following properties:*

(i) *They are continuous and satisfy the terminal condition* $v(T, \cdot) \leq g(\cdot)$ *together with the growth condition*

$$|v(t,x)| \leq C(v)\psi(x), 0 \leq t \leq T,\ x \in \mathcal{O}, \quad \text{for some} \ \ C(v) < \infty. \tag{3.3}$$

(ii) *There exists a bound* $L(v) < \infty$*, depending on* v*, such that for each stopping time* τ *and each* $\xi \in \mathcal{F}_\tau$ *such that* $\mathbb{P}(\xi \in \mathcal{O}) = 1$ *and* $\mathbb{E}[\psi(\xi)] < \infty$*, there exists a control* $u \in \mathcal{U}_{\tau,T}$ *defined on* $[\tau, T]$ *adapted to* $\mathbb{F}$*, satisfying the bound* $\|u\| \leq L(v)$ *and such that for any* $\mathbb{F}$*-stopping time* $\rho \in [\tau, T]$ *we have that*

$$v(\tau, \xi) \leq \mathbb{E}\left[v\left(\rho, X_\rho^{u;\tau,\xi}\right) \middle| \mathcal{F}_\tau \right] \ \ a.s. \tag{3.4}$$

We do not expect the value function to be a stochastic sub-solution, except in the situations when there exists and admissible optimal control. As already mentioned, this is rarely the case, with our very restrictive definition of admissibility. However, this does not cause any problem in the course of completing the Stochastic Perron Method: while the value function is not a sub-solution itself, it can be approximated by sub-solutions.

Remark 3.2. *We ask for the sub-martingale property to hold only in between the fixed stopping time* τ *and any later* $\rho \geq \tau$*, which is actually less than the full sub-martingale property on the stochastic interval* $[\tau, T]$.

Assumption 3.4. *We assume that* $\mathcal{V}^- \neq \emptyset$.

Remark 3.3. *Assumption 3.4 is satisfied, for example, when* g *is bounded from below.*

A crucial property of the set of stochastic solutions is the following stability result:

Proposition 3.1. *If* v^1 *and* v^2 *are two stochastic sub-solutions, then* $v = v^1 \vee v^2$ *is also a stochastic sub-solution.*

Proof. We will only show that v satisfies item (ii) of the definition of stochastic sub solution. We can choose the uniform bound corresponding to v as

$$L(v) = L(v^1) \vee L(v^2).$$

Now, fix a stopping time τ and a random variable $\xi \in \mathcal{F}_\tau$ with $\mathbb{P}(\xi \in \mathcal{O}) = 1$ and $\mathbb{E}[\psi(\xi)] < \infty$. Then, by the definition of the stochastic sub-solutions v^1 and v^2, it follows that there are two controls $\|u_1\| \leq L(v^1)$ and $\|u_2| \leq L(v^2)$ satisfying

$$v^i(\tau, \xi) \leq \mathbb{E}[v^i(\rho, X_\rho^{u_i;\tau,\xi})|\mathcal{F}_\tau], \quad i \in \{1, 2\}.$$

Now define a control u (on the stochastic interval $(\tau, T]$) by

$$u = 1_{\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\}} u_1 + 1_{\{v^1(\tau,\xi) < v^2(\tau,\xi)\}} u_2. \tag{3.5}$$

Now, for each $\tau \leq \rho \leq T$, we have

(1) on $\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\} \in \mathcal{F}_\tau$ we have

$$v^1(\rho, X_\rho^{u_1;\tau,\xi}) = v^1(\rho, X_\rho^{u;\tau,\xi}) \leq v(\rho, X_\rho^{u;\tau,\xi});$$

(2) on $\{v^1(\tau,\xi) < v^2(\tau,\xi)\} \in \mathcal{F}_\tau$ we have

$$v^2(\rho, X_\rho^{u_2;\tau,\xi}) = v^2(\rho, X_\rho^{u;\tau,\xi}) \leq v(\rho, X_\rho^{u;\tau,\xi}).$$

Applying the definition of sub-solutions for v^1 and v^2 (for controls u_1 and u_2) we get

$$1_{\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\}} v^1(\tau,\xi) \leq \mathbb{E}\left[1_{\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\}} v^1(\rho, X_\rho^{u;\tau,\xi}) \middle| \mathcal{F}_\tau \right] \ a.s.,$$

since $\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\} \in \mathcal{F}_\tau$. Therefore, according to item (1) above we have

$$1_{\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\}} v^1(\tau,\xi) \leq \mathbb{E}\left[1_{\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\}} v(\rho, X_\rho^{u;\tau,\xi}) \middle| \mathcal{F}_\tau \right] \ a.s. \tag{3.6}$$

Similarly, we obtain

$$1_{\{v^1(\tau,\xi) < v^2(\tau,\xi)\}} v^2(\tau,\xi) \leq \mathbb{E}\left[1_{\{v^1(\tau,\xi) < v^2(\tau,\xi)\}} v^2(\rho, X_\rho^{u;\tau,\xi}) \middle| \mathcal{F}_\tau \right] \ a.s.,$$

and by item (2) above we have

$$1_{\{v^1(\tau,\xi) < v^2(\tau,\xi)\}} v^2(\tau,\xi) \leq \mathbb{E}\left[1_{\{v^1(\tau,\xi) < v^2(\tau,\xi)\}} v(\rho, X_\rho^{u;\tau,\xi}) \middle| \mathcal{F}_\tau \right] \ a.s. \tag{3.7}$$

Putting (3.6) and (3.7) together we conclude. □

Theorem 3.1. (The supremum of stochastic sub-solutions is a viscosity super-solution) *Let Assumptions 2.1-(1), 3.1, 3.2, 3.3 and 3.4 hold true. Assume also that g is a lower semi-continuous function and $V < \infty$. Then the lower stochastic envelope*

$$v^- := \sup_{v \in \mathcal{V}^-} v \leq V < \infty,$$

is a viscosity super-solution of (2.2). *Moreover, if we define*

$$v^-(T-, x) := \liminf_{(t'<T, x') \to (T,x)} v^-(t', x'), \quad x \in \mathcal{O}, \tag{3.8}$$

then the function $v^-(T-, \cdot) (\geq g(\cdot))$ is a viscosity super-solution of (2.3).

Remark 3.4. *The function v^- may not have a limit from the left at $t = T$. We, therefore, modify this function as described in* (3.8) *at $t = T$. If we consider the function v^- with the new terminal condition $v^-(T-, \cdot)$, it still is lower-semi continuous, as it is used in the proof of Theorem 5.1.*

Proof.
Step 1. The fact that $v^- \leq V$ follows directly from the definition of the class of stochastic sub-solutions and by the definition of $\mathcal{U}_{t,T}$ and V.

Step 2. *The interior super-solution property.* Let $\varphi : [0,T] \times \mathcal{O} \to \mathbb{R}^d$ be a $C^{1,2}$-test function such that $v^- - \varphi$ attains a strict local minimum equal to zero at some parabolic interior point $(t_0, x_0) \in [0,T) \times \mathcal{O}$. We first prove that

$$-\varphi_t(t_0,x_0) - H(t_0,x_0,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0)) \geq 0, \tag{3.9}$$

by contradiction. To this end, assume that

$$(-\varphi_t - \sup_u L_t^u \varphi)(t_0,x_0) < 0.$$

But then there exists $\tilde{u} \in U$ such that

$$(-\varphi_t - L_t^{\tilde{u}} \varphi)(t_0,x_0) < 0. \tag{3.10}$$

Since the coefficients of the SDE are continuous there exists a small enough ball $B(t_0,x_0,\varepsilon)$ such that

$$-\varphi_t - L_t^{\tilde{u}} \varphi(t,x) < 0, \quad (t,x) \in B(t_0,x_0,\varepsilon),$$

and

$$\varphi(t,x) < v^-(t,x), \quad (t,x) \in B(t_0,x_0,\varepsilon) - \{(t_0,x_0)\}.$$

To be precise, all along the paper, we use the norm $\|(t,x)\| = \max\{|t|,|x|\}$, so

$$B(t_0,x_0,\varepsilon) := \{(t,x) \in [0,T) \times \mathcal{O} | \max\{|t-t_0|,|x-x_0|\} < \varepsilon\}.$$

Since $v^- - \varphi$ is lower semi-continuous and $\overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2)$ is compact, there exists a $\delta > 0$ satisfying

$$v^- - \delta \geq \varphi \quad \text{on } \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2).$$

Using Proposition 4.1 in [2] together with Proposition 3.1 above, we obtain a (countable) increasing sequence of stochastic sub-solutions $v_n \nearrow v^-$. Now, since φ is continuous, as well as v_n's, we can use a Dini argument (identical to the one in Lemma 2.4 of [1]) to conclude that for $\delta' \in (0,\delta)$ there exists a stochastic sub-solution $v = v_n$ (for some large enough n) such that

$$v - \delta' \geq \varphi \quad \text{on } \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2). \tag{3.11}$$

Choosing $\eta \in (0,\delta')$ small enough we have that the function

$$\varphi^\eta := \varphi + \eta$$

satisfies

$$-\varphi_t^\eta - L_t^{\tilde{u}} \varphi^\eta(t,x) < 0, \quad (t,x) \in B(t_0,x_0,\varepsilon),$$
$$\varphi^\eta(t,x) < v(t,x), \quad (t,x) \in \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2),$$

and

$$\varphi^\eta(t_0,x_0) = v^-(t_0,x_0) + \eta > v^-(t_0,x_0).$$

Now we define

$$v^\eta = \begin{cases} v \vee \varphi^\eta \text{ on } \overline{B(t_0,x_0,\varepsilon)}, \\ v \text{ outside } \overline{B(t_0,x_0,\varepsilon)}. \end{cases}$$

Clearly, v^η is continuous and $v^\eta(t_0,x_0) = \varphi^\eta(t_0,x_0) > v^-(t_0,x_0)$. And since ε can be chosen so that $T > t_0 + \varepsilon$, v^η satisfies the terminal condition. In addition, the growth condition in (i) Definition 3.2 holds for v^η, since such growth condition holds for the approximate supremum v (although we may not have, without additional assumptions, a similar growth condition on v^-).

We only need to show that v^η satisfies (ii) in Definition 3.2 to get a contradiction and complete the proof. Let $0 \leq \tau \leq T$ be a fixed stopping time and $\xi \in \mathcal{F}_\tau$,

$\mathbb{P}(\xi \in \mathcal{O}) = 1$, such that $\mathbb{E}[\psi(\xi)] < \infty$. We need to construct a control $u \in \mathcal{U}_{\tau,T}$ that works for v^η in (ii) in Definition 3.2. Following the arguments in the proof of Proposition 3.1, such a control u can be constructed in a surprisingly simple way, *which represents a significant technical improvement over the previous work* [2] *or* [1].

Denote by $u_0 \in \mathcal{U}_{\tau,T}$ the control corresponding to initial time τ and initial condition ξ in (ii) in Definition 3.2 for the stochastic sub-solution v. Denote by A the event

$$A = \{(\tau, \xi) \in B(t_0, x_0, \varepsilon/2) \text{ and } \varphi^\eta(\tau, \xi) > v(\tau, \xi)\}.$$

Recalling (3.10), we define the new admissible control $u_1 \in \mathcal{U}_{\tau,T}$ by

$$u_1 = \tilde{u} \times 1_A + u_0 \times 1_{A^c},$$

and by τ_1 the first time after τ when the diffusion started at ξ and controlled by u_1 hits the boundary of $B(t_0, x_0, \varepsilon/2)$:

$$\tau_1 = \inf\{\tau \le t \le T | X_t^{u_1;\tau,\xi} \in \partial B(t_0, x_0, \varepsilon/2)\}.$$

Now, denote by

$$\xi_1 = X_{\tau_1}^{u_1;\tau,\xi} \in \partial B(t_0, x_0, \varepsilon/2),$$

and by $u_2 \in \mathcal{U}_{\tau_1,T}$ the control in (ii) in Definition 3.2 corresponding to v for the starting time τ_1 and initial condition ξ_1. Now, we can finally define

$$u = u_1 \times 1_{\{\tau < t \le \tau_1\}} + u_2 \times 1_{\{\tau_1 < t \le T\}}.$$

Note that the control u is bounded by $L(v) \vee |\tilde{u}|$, and, therefore, it is admissible. Consider any stopping time ρ such that $\tau \le \rho \le T$. On the event A, $\varphi^\eta(\cdot, X_\cdot)$ is a sub-martingale up to $\rho \wedge \tau_1$ (because of Itô's formula together with the fact that φ^η is bounded in the interior ball), which reads

$$1_A \varphi^\eta(\tau, \xi) \le \mathbb{E}[1_A \varphi^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{\tilde{u};\tau,\xi}) | \mathcal{F}_\tau] \ a.s.$$

Since

$$1_A \varphi^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{\tilde{u};\tau,\xi}) = 1_A \varphi^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{u;\tau,\xi}) \le 1_A v^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{u;\tau;\xi}),$$

we actually obtain

$$1_A v^\eta(\tau, \xi) = 1_A \varphi^\eta(\tau, \xi) \le \mathbb{E}[1_A v^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{u;\tau,\xi}) | \mathcal{F}_\tau] \ a.s.$$

Next, we use the fact that u_1 is the "optimal" control for v, together with $v = v^\eta$ everywhere outside the open ball $B(t_0, x_0, \varepsilon/2)$, to obtain:

$$1_{A^c} v^\eta(\tau, \xi) = 1_{A^c} v(\tau, \xi) \le \mathbb{E}[1_{A^c} v(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{u_1;\tau.\xi}) | \mathcal{F}_\tau] = \mathbb{E}[1_{A^c} v^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{u;\tau,\xi}) | \mathcal{F}_\tau].$$

Putting the above together, we obtain:

$$v^\eta(\tau, \xi) \le \mathbb{E}\left[v^\eta\left(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{u;\tau,\xi}\right) \middle| \mathcal{F}_\tau \right] \ a.s. \tag{3.12}$$

Let us introduce yet another notation: $B = \{\rho \le \tau_1\} \in \mathcal{F}_{\tau_1}$. We know that, on the boundary $\partial B(t_0, x_0, \varepsilon/2)$, $v = v^\eta$. Applying the definition of u, together with the fact that u_2 is "optimal" for v starting at τ_1 with condition ξ_1, we have

$$1_{B^c} v^\eta(\tau_1, \xi_1) = 1_{B^c} v(\tau_1, \xi_1) \le \mathbb{E}[1_{B^c} v(\rho, X_\rho^{u_2;\tau_1,\xi_1}) | \mathcal{F}_{\tau_1}] \le \mathbb{E}[1_{B^c} v^\eta(\rho, X_\rho^{u;\tau,\xi}) | \mathcal{F}_{\tau_1}].$$

If we rewrite the RHS in (3.12) as

$$\mathbb{E}\left[v^\eta\left(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{u;\tau,\xi}\right) \middle| \mathcal{F}_\tau \right] = \mathbb{E}\left[1_B v^\eta\left(\rho, X_\rho^{u;\tau,\xi}\right) + 1_{B^c} v^\eta(\tau_1, \xi_1) \middle| \mathcal{F}_\tau \right],$$

and use the tower property, we get, indeed

$$v^\eta(\tau,\xi) \le \mathbb{E}\left[v^\eta\left(\rho, X_\rho^{u;\tau,\xi}\right) \middle| \mathcal{F}_\tau\right] \ a.s.$$

This completes the proof of (3.9), from which it follows that

$$H(t_0,x_0,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0)) < \infty.$$

Thanks to Assumption 2.1-(1) we also have that

$$G(t_0,x_0,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0)) \ge 0, \tag{3.13}$$

finishing the proof of interior super-solution property.

Step 3. *The terminal condition, Part I.* We will show that $v^-(T,\cdot) = g(\cdot)$. Assume that for some $x_0 \in \mathcal{O}$ we have

$$v^-(T,x_0) < g(x_0).$$

We will use this information to construct a contradiction. Since $g(\cdot)$ is lower-semi continuous then there exists an $\varepsilon > 0$ such that

$$g(x) \ge v^-(T,x_0) + \varepsilon, \quad \text{if} \quad |x - x_0| \le \varepsilon.$$

Due to the fact that v^- is lower-semi continuous, it is bounded from below on the compact set

$$(\overline{B(T,x_0,\varepsilon)} - B(T,x_0,\varepsilon/2)) \cap ([0,T]\times\mathcal{O}).$$

For a small enough $\eta > 0$ we have that

$$v^-(T,x_0) - \frac{\varepsilon^2}{4\eta} < -\varepsilon + \inf_{(t,x)\in(\overline{B(T,x_0,\varepsilon)} - B(T,x_0,\varepsilon/2))\cap([0,T]\times\mathcal{O})} v^-(t,x).$$

Since the above inequality is strict, following the proof of Step 2 in Theorem 3.1, we use again Proposition 4.1 in [2] together with Proposition 3.1 above, and a Dini argument to find a stochastic sub-solution $v \in \mathcal{V}^-$ such that

$$v^-(T,x_0) - \frac{\varepsilon^2}{4\eta} < -\varepsilon + \inf_{(t,x)\in(\overline{B(T,x_0,\varepsilon)} - B(T,x_0,\varepsilon/2))\cap([0,T]\times\mathcal{O})} v(t,x). \tag{3.14}$$

For $k > 0$ define

$$\varphi^{\eta,\varepsilon,k}(t,x) = v^-(T,x_0) - \frac{|x-x_0|^2}{\eta} - k(T-t).$$

Choose k large enough, at least as large as $k \ge \varepsilon/4\eta$ but possibly much larger, such that

$$\left[-\varphi_t^{\eta,\varepsilon,k} - \sup_u L_t^u \varphi^{\eta,\varepsilon,k}\right](t_0,x_0) < 0 \quad \text{on} \quad \overline{B(T,x_0,\varepsilon)}.$$

Using (3.14) we obtain

$$\varphi^{\eta,\varepsilon,k} \le -\varepsilon + v \quad \text{on} \quad (\overline{B(T,x_0,\varepsilon)} - B(T,x_0,\varepsilon/2)) \cap ([0,T]\times\mathcal{O}).$$

On the other hand,

$$\varphi^{\eta,\varepsilon,k}(T,x) \le v^-(T,x_0) \le g(x) - \varepsilon, \quad \text{for} \quad |x-x_0| \le \varepsilon.$$

Now, let $\delta < \varepsilon$ and define

$$v^{\varepsilon,\eta,k,\delta} := \begin{cases} v \vee (\varphi^{\varepsilon,\eta,k} + \delta) & \text{on} \quad \overline{B(T,x_0,\varepsilon)}, \\ v \quad \text{outside} & \overline{B(T,x_0,\varepsilon)}. \end{cases}$$

Now using the idea in Step 1 of the proof, we can show that $v^{\varepsilon,\eta,k,\delta} \in \mathcal{V}^-$ but $v^{\varepsilon,\eta,k,\delta}(T,x_0) = v^-(T,x_0) + \delta > v^-(T,x_0)$, leading to a contradiction.

The only reason we actually proved $v^-(T,\cdot) = g(\cdot)$ was to get some information about the left liminf $v^-(T-,\cdot)$. More precisely, since v^- is lower semi-continuous, we know that

$$g(\cdot) = v^-(T,\cdot) \leq v^-(T-,\cdot).$$

In order to finish the proof of the Theorem, we only need to show that $v^-(T-,\cdot)$ is a viscosity super-solution of (2.3), which we will do in the next step.

Step 4. *The terminal condition, Part II.* We show that the l.s.c. function $v^-(T-,\cdot)$ is a viscosity super-solution of

$$G(T,x,v^-_x(T,x),v^-_{xx}(T,x)) \geq 0, \quad x \in \mathcal{O}.$$

The arguments used below trace back to [4] and were technically refined later for more general models of super-hedging in [7], [18] and others, as presented in the survey paper [17]. We basically use the notation from Lemma 4.3.2 in [16] which summarizes the existing literature.

More precisely, we rely on the fact that v^- satisfies the same equation in the interior, a fact we established in Step 2, to get information about the limit as $t \to T$. Let $y \in \mathbb{R}^d$ and $\psi(x)$ be a test function satisfying

$$0 = v^-(T-,y) - \psi(y) = \min_{x\in\mathbb{R}^d}(v^-(T-,x) - \psi(x)). \tag{3.15}$$

By the very definition of $v^-(T-,\cdot)$, there exists a sequence (s_m,y_m) converging to (T,y) with $s_m < T$ such that

$$\lim_{m\to\infty} v^-(s_m,y_m) = v^-(T-,y).$$

Let us construct another test function that depends both on t and x variables:

$$\psi_m(t,x) = \psi(x) - |x-y|^4 + \frac{T-t}{(T-s_m)^2},$$

and choose $(t_m,x_m) \in [s_m,T]\times\overline{B(y,\varepsilon)}$ as a minimum of $v^- - \psi_m$ on $[s_m,T]\times\overline{B(y,\varepsilon)}$ where ε is chosen small enough so that $\overline{B(y,\varepsilon)} \subset \mathcal{O}$.

What we would like to do next is to show that in fact $t_m < T$ for large enough m and that $x_m \to y$. The first fact follows from the observation that

$$v^-(s_m,y_m) - \psi_m(s_m,y_m) \leq -\frac{1}{2(T-s_m)} < 0,$$

for large enough m and that

$$v^-(T-,x) - \psi_m(T,x) \geq v^-(T-,x) - \psi(x) \geq 0,$$

where the second inequality follows from (3.15). Let us focus on the convergence of x_m to y. The sequence (x_m) converges (up to choosing a sub-sequence) to some $z \in \overline{B(y,1)}$. By construction, $s_m \leq t_m$. Using this and the choice of (t_m,x_m) we obtain the following string of inequalities:

$$\begin{aligned}0 &\le (v^-(T-,z)-\psi(z))-(v^-(T-,y)-\psi(y))\\ &\le \liminf_{m\to\infty}\Big[(v^-(t_m,x_m)-\psi(x_m))-(v^-(s_m,y_m)-\psi(y_m))\Big]\\ &\le \liminf_{m\to\infty}\Big[(v^-(t_m,x_m)-\psi_m(t_m,x_m))-(v^-(s_m,y_m)-\psi_m(s_m,y_m))\\ &-|x_m-y|^4+\frac{T-t_m}{(T-s_m)^2}+|y_m-y|^4-\frac{T-s_m}{(T-s_m)^2}\Big]\\ &\le \liminf_{m\to\infty}\Big[-|x_m-y|^4+|y_m-y|^4\Big]=-|z-y|^4,\end{aligned}$$

which proves that $z=y$.

We know that (t_m,x_m) is a minimizer of $v^- - \psi_m$ over $[s_m,T]\times\overline{B(y,\varepsilon)}$ by definition, and we also know that $s_m\le t_m<T$ for large m. Since $x_m\to y$, we conclude that (for m large enough) we have $(v^- - \psi_m)(t_m,x_m)\le(v^- - \psi_m)(t,x)$ for $t_m\le t<T,\ |x-x_m|\le\varepsilon/2$. While this does not mean that (t_m,x_m) is a local interior min for $v^- - \psi_m$ (because we may have $t_m=s_m$), it does mean that we have a local "parabolic interior minimum". It is well known that, for example from [6], for parabolic equations, a "parabolic interior minimum" is enough to use ψ_m as a test function at (t_m,x_m), and, therefore first conclude that

$$-D_t\psi_m(t_m,x_m)-H(t_m,x_m,D_x\psi_m(t_0,x_0),D_x^2\psi_m(t_m,x_m))\ge 0,$$

so $H(t_m,x_m,D_x\psi_m(t_0,x_0),D_x^2\psi_m(t_m,x_m))<\infty$ and, consequently,

$$G(t_m,x_m,D_x\psi_m(t_m,x_m),D_x^2\psi_m(t_m,x_m))\ge 0.$$

Now the claim of this step follows from the continuity of G and the fact that $x_m\to y$, as the derivatives of ψ_m with respect to x converge to those of ψ.

□

4. Stochastic super-solutions

In this section we consider the weak formulation of the stochastic control problem.

Assumption 4.1. *We assume that the coefficients* $b:[0,T]\times\mathbb{R}^d\times U\to\mathbb{R}^d$ *and* $\sigma:[0,T]\times\mathbb{R}^d\times U\to\mathbb{M}_{d,d'}(\mathbb{R})$ *are continuous.*

Definition 4.1. *For each* (s,x) *we denote by* $\mathcal{U}_{s,x}$ *the set of weak admissible controls for the* (2.1), *by which we mean a*

$$\Big(\Omega^{s,x},\mathcal{F}^{s,x},(\mathcal{F}_t^{s,x})_{s\le t\le T},\mathbb{P}^{s,x},(W_t^{s,x})_{s\le t\le T},(X_t^{s,x})_{s\le t\le T},(u_t)_{s\le t\le T}\Big),$$

where

1. $(\Omega^{s,x},\mathcal{F}^{s,x},(\mathcal{F}_t^{s,x})_{s\le t\le T},\mathbb{P}^{s,x})$ *is an arbitrary stochastic basis satisfying the usual conditions,*
2. $W^{s,x}$ *is a* d'*-dimensional Brownian motion with respect to the filtration* $(\mathcal{F}_t^{s,x})_{s\le t\le T}$,
3. u *is a predictable and uniformly bounded* U*-valued process,*

(4) *$X^{s,x}$ is a continuous and adapted process satisfying* (2.1) *with initial condition $X_s = x \in \mathcal{O}$, and $\mathbb{P}^{s,x}(X^{s,x}_t \in \mathcal{O},\ s \leq t \leq T) = 1$ together with*

$$\mathbb{E}^{s,x}\left[\sup_{s \leq t \leq T} \psi(X^{s,x}_t)\right] < \infty,$$

for the gauge function ψ in Section 3.

Now, for some measurable function $g : \mathcal{O} \to \mathbb{R}$, we denote by

$$\mathfrak{V}(s,x) := \sup_{\mathcal{U}^{s,x}} \mathbb{E}^{s,x}[g(X^{s,x}_T)], \tag{4.1}$$

the value function of the weak control problem.

Assumption 4.2. *The pay-off function g is an upper semi-continuous function satisfying $|g(\cdot)| \leq C\psi(\cdot)$.*

Remark 4.1. (1) *Because of the growth assumption on weakly controlled solutions, $\mathbb{E}^{s,x}[g(X^{s,x}_T)]$ is well defined and finite, so $\mathfrak{V} > -\infty$.*
(2) *When both are well-defined it clearly holds that $V \leq \mathfrak{V}$.*

Our goal in this section is to construct an upper bound of $\mathfrak{V}$ that is a viscosity sub-solution.

Definition 4.2. *The set of stochastic super-solutions for the parabolic PDE* (2.2), *denoted by $\mathcal{V}^+$, is the set of functions $v : [0,T] \times \mathcal{O}^d \to \mathbb{R}$ which have the following properties:*

(1) *They are continuous and satisfy the terminal condition $v(T,\cdot) \geq g(\cdot)$ together with the growth condition*

$$|v(t,x)| \leq C(v)\psi(x), 0 \leq t \leq T,\ x \in \mathcal{O}.$$

(2) *For each $(s,x) \in [0,T] \times \mathcal{O}$, and each weak control*

$$\Big(\Omega^{s,x}, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{s \leq t \leq T}, \mathbb{P}^{s,x}, (W^{s,x}_t)_{s \leq t \leq T}, (X^{s,x}_t)_{s \leq t \leq T}, (u_t)_{s \leq t \leq T}\Big),$$

the process $(u(t, X^{s,x}_t))_{s \leq t \leq T}$ is a super-martingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathcal{F}^{s,x}_t)_{s \leq t \leq T}$.

Assumption 4.3. *$\mathcal{V}^+ \neq \emptyset$.*

Remark 4.2. *Assumption 4.3 is satisfied, for example, when g is bounded from above.*

Theorem 4.1. (The infimum of stochastic super-solutions is a viscosity sub-solution) *Let Assumptions 2.1-(2), 4.1, 4.2, and 4.3 hold true. Then $v^+ = \inf_{v \in \mathcal{V}^+} v$ is a viscosity sub-solution of* (2.2). *Moreover, the USC function $v^+(T,\cdot)$ is a viscosity sub-solution of* (2.3).

Proof.
Step 1. The fact that $v^+ \geq \mathfrak{V}$ follows directly from the definition of the class of stochastic sub-solutions and by the definition of $\mathcal{U}$.
Step 2. *The interior sub-solution property.* Let $\varphi : [0,T] \times \mathcal{O} \to \mathbb{R}^d$ be a $C^{1,2}$-test function such that $v^+ - \varphi$ attains a strict local maximum equal to zero at some parabolic interior point $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$, where the viscosity sub-solution property fails, i.e.,

$$\min\{-\varphi_t(t_0,x_0) - H(t,x,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0)), G(t,x,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0))\} > 0.$$

Then since G is continuous and H is continuous in the interior of its domain it follows that there exists a small enough ball $B(t_0,x_0,\varepsilon)$ such that, for all $(t,x) \in B(t_0,x_0,\varepsilon)$ we have:

$$\min\{-\varphi_t(t,x) - H(t,x,\varphi_x(t,x),\varphi_{xx}(t,x)), G(t,x,\varphi_x(t,x),\varphi_{xx}(t,x))\} > 0.$$

Now the rest of the proof of this step is very similar to the corresponding step in the proof of Theorem 2.1 in [2], but much simplified by following the stopping idea in the proof of Theorem 3.1 (step 2) above. For the sake of completeness and the convenience of the reader we actually include the remaining part of the proof. The function $v^+ - \varphi$ is uper semi-continuous and $\overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2)$ is compact, there exists a $\delta > 0$ satisfying

$$v^+ + \delta \leq \varphi \quad \text{on } \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2).$$

Using Proposition 4.1 in [2] together with the obvious observation that the minimum of two stochastic super-solutions is also a stochastic super-solution, we obtain a (countable) decreasing sequence of stochastic super-solutions $v_n \searrow v^+$. Now, since φ is continuous, as well as v_n's, we can use once again a Dini argument (identical to the one in Lemma 2.4 of [1]) to conclude that for $\delta' \in (0,\delta)$ there exists a stochastic super-solution $v = v_n$ (for some large enough n) such that

$$v + \delta' \leq \varphi \quad \text{on } \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2).$$

Choosing $\eta \in (0,\delta')$ small enough we have that the function

$$\varphi^\eta := \varphi - \eta$$

satisfies

$$-\varphi^\eta_t(t,x) - H(t,x,\varphi^\eta_x(t,x),\varphi^\eta_{xx}(t,x)) > 0, (t,x) \in B(t_0,x_0,\varepsilon),$$
$$\varphi^\eta(t,x) > v(t,x),\ (t,x) \in \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2),$$

and

$$\varphi^\eta(t_0,x_0) = v^+(t_0,x_0) - \eta < v^+(t_0,x_0).$$

Now we define, similarly to Step 2 above,

$$v^\eta = \begin{cases} v \wedge \varphi^\eta \text{ on } \overline{B(t_0,x_0,\varepsilon)}, \\ v \text{ outside } \overline{B(t_0,x_0,\varepsilon)}. \end{cases}$$

Clearly, v^η is continuous and $v^\eta(t_0,x_0) = \varphi^\eta(t_0,x_0) > v^-(t_0,x_0)$. And since ε can be chosen so that $T > t_0 + \varepsilon$, v^η satisfies the terminal condition. Again, the growth condition in (i) Definition 3.2 holds for v^η, since such growth condition holds for the approximate infimum v. We now only need to show that v^η satisfies (ii) in Definition 4.2 to get a contradiction and complete the proof. Fix an admissible weak control

$$\Big(\Omega^{s,x}, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{s\leq t\leq T}, \mathbb{P}^{s,x}, (W^{s,x}_t)_{s\leq t\leq T}, (X^{s,x}_t)_{s\leq t\leq T}, (u_t)_{s\leq t\leq T}\Big).$$

Fix now $s \leq \tau \leq \rho \leq T$ two stopping times of the filtration $(\mathcal{F}^{s,x}_t)_{s\leq t\leq T}$. Denote, similarly to Step 2, by A the event

$$A = \{(\tau, X^{s,x}_\tau) \in B(t_0,x_0,\varepsilon/2) \text{ and } \varphi^\eta(\tau, X^{s,x}_\tau) < v(\tau, X^{s,x}_\tau)\}.$$

Denote by τ_1 the first time after τ when the diffusion hits the boundary of $B(t_0,x_0,\varepsilon/2)$:

$$\tau_1 = \inf\{\tau \leq t \leq T | X^{s,x}_t \in \partial B(t_0,x_0,\varepsilon/2)\}.$$

On the event A, $\varphi^\eta(\cdot, X_\cdot^{s,x})$ is a continuous super-martingale up to $\rho \wedge \tau_1$ (because of Itô's formula together with the fact that φ^η is bounded in the interior ball), which reads

$$1_A \varphi^\eta(\tau, X_\tau^{s,x}) \geq \mathbb{E}^{s,x}[1_A \varphi^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{s,x})|\mathcal{F}_\tau^{s,x}]\ \mathbb{P}^{s,x} - a.s.$$

Since $1_A \varphi^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{s,x}) \geq 1_A v^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{s,x})$, we have

$$1_A v^\eta(\tau, X_\tau^{s,x}) = 1_A \varphi^\eta(\tau, X_\tau^{s,x}) \geq \mathbb{E}^{s,x}[1_A v^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{s,x})|\mathcal{F}_\tau^{s,x}]\ \mathbb{P}^{s,x} - a.s.$$

Next, we use the optional sampling theorem applied to the continuous super-martingale $v(\cdot, X_\cdot^{s,x})$ in between the stopping times $\tau \leq \rho \wedge \tau_1$, together with the observation that $v = v^\eta$ everywhere outside the open ball $B(t_0, x_0, \varepsilon/2)$, to obtain:

$$\begin{aligned} 1_{A^c} v^\eta(\tau, X_\tau^{s,x}) = 1_{A^c} v(\tau, X_\tau^{s,x}) &\geq \mathbb{E}^{s,x}[1_{A^c} v(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{s,x})|\mathcal{F}_\tau^{s,x}] \\ &\geq \mathbb{E}^{s,x}[1_{A^c} v^\eta(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{s,x})|\mathcal{F}_\tau^{s,x}],\ \mathbb{P}^{s,x} - a.s. \end{aligned}$$

Putting the above together, we obtain:

$$v^\eta(\tau, X_\tau^{s,x}) \geq \mathbb{E}^{s,x}\left[v^\eta\left(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{s,x}\right)\middle|\mathcal{F}_\tau^{s,x}\right]\ \mathbb{P}^{s,x} - a.s. \tag{4.2}$$

Let us again introduce the notation: $B = \{\rho \leq \tau_1\} \in \mathcal{F}_{\tau_1\wedge\rho}^{s,x}$. We know that, on the boundary $\partial B(t_0, x_0, \varepsilon/2)$, $v = v^\eta$. Together with the optional sampling theorem applied to the continuous super-martingale $v(\cdot, X_\cdot^{s,x})$ between $\tau_1 \wedge \rho$ and ρ we have

$$\begin{aligned} 1_{B^c} v^\eta(\tau_1, X_{\tau_1}^{s,x}) = 1_{B^c} v(\tau_1, X_{\tau_1}^{s,x}) &\geq \mathbb{E}^{s,x}[1_{B^c} v(\rho, X_\rho^{s,x})|\mathcal{F}_{\tau_1}^{s,x}] \\ &\geq \mathbb{E}[1_{B^c} v^\eta(\rho, X_\rho^{s,x})|\mathcal{F}_{\tau_1}^{s,x}],\ \mathbb{P}^{s,x} - a.s. \end{aligned}$$

We finally rewrite the RHS in (4.2) as

$$\mathbb{E}^{s,x}\left[v^\eta\left(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{s,x}\right)\middle|\mathcal{F}_\tau^{s,x}\right] = \mathbb{E}^{s,x}\left[1_B v^\eta\left(\rho, X_\rho^{s,x}\right) + 1_{B^c} v^\eta(\tau_1, X_{\tau_1}^{s,x})\middle|\mathcal{F}_\tau^{s,x}\right],$$

and use the tower property to obtain

$$v^\eta(\tau, X_\tau^{s,x}) \geq \mathbb{E}^{s,x}\left[v^\eta\left(\rho, X_\rho^{s,x}\right)\middle|\mathcal{F}_\tau^{s,x}\right]\ \mathbb{P}^{s,x} - a.s.$$

Since this happens for any stopping times $s \leq \tau \leq \rho \leq T$ of the filtration $(\mathcal{F}_t^{s,x})_{s\leq t\leq T}$, we have, indeed, that v^η is a stochastic super-solution, leading to a contradiction and completing the proof.

Step 3. *The boundary condition.*

Let $x_0 \in \mathcal{O}$ and ψ be a smooth function on $\mathcal{O}$ such that

$$0 = v^+(T, x_0) - \psi(x_0) = \max_{\mathcal{O}}(v^+(T, x) - \psi(x)).$$

Assume, in addition, without losing generality, that the maximum is strict. Let us assume, by contradiction, that

$$G(T, x_0, \psi_x(x_0), \psi_{xx}(x_0)) > 0 \text{ and } v^+(T, x_0) > g(x_0). \tag{4.3}$$

Since G is continuous, and, in addition, G is finite *and* continuous in the open set $G > 0$, we conclude that, there exists small $\varepsilon, \delta_0 > 0$ and a finite constant C such that

$$H(t, x, \psi_x(x), \psi_{xx}(x)) < C, \quad T - t \leq \delta_0, |x - x_0| \leq \varepsilon.$$

In addition, we also have (for small enough ε)

$$\psi(x) \geq g(x) + \varepsilon, \quad |x - x_0| \leq \varepsilon.$$

Now, the whole idea is based on constructing a local super-solution

$$\psi^k(t,x) = \psi(x) + k(T-t)$$

for large k, *by decoupling the bounds δ and ε in the estimate above*, then pushing it slightly down. Namely, we will make δ much smaller than ε. Fix δ_0 and ε as above. Denote by

$$h(\delta) = \sup_{T-t\leq\delta, \frac{\varepsilon}{2}\leq|x-x_0|\leq\varepsilon} \Big(v^+(t,x) - \psi(x)\Big),\ 0<\delta<\delta_0.$$

Interpreting ψ as a continuous function of two variables (t,x), which actually does not depend on t and taking into account that v^+ is USC, there exist a point where the maximum above is attained, i.e.

$$h(\delta) = v^+(t_\delta, x_\delta) - \psi(x_\delta).$$

By compactness, we can subtract a sub-sequence (we still denote it as $\delta \searrow 0$) such that

$$(t_\delta, x_\delta) \to (T, x^*),\ \ \frac{\varepsilon}{2} \leq |x^* - x_0| \leq \varepsilon.$$

Since v^+ is USC, we conclude that

$$\begin{aligned} \limsup_{\delta\searrow 0} h(\delta) &= \limsup_{\delta\searrow 0} \Big(v^+(t_\delta, x_\delta) - \psi(x_\delta)\Big) \\ &\leq v^+(T,x^*) - \psi(x^*) \leq \sup_{\frac{\varepsilon}{2}\leq|x-x_0|\leq\varepsilon} \Big(v^+(T,x) - \psi(x)\Big) < 0, \end{aligned} \tag{4.4}$$

where the last inequality follows from the fact that we have a strict max at x_0 and the last supremum is actually attained. Therefore, we can choose $\delta < \delta_0$ small enough such that $h(\delta) < 0$. Now, for *this* fixed δ, with the notation

$$\delta' = -h(\delta) > 0$$

we have

$$v^+(t,x) \leq \psi(x) - \delta',\ \ T-t\leq\delta, \frac{\varepsilon}{2} \leq |x-x_0| \leq \varepsilon. \tag{4.5}$$

Denote by D the compact "rectangular donut"

$$D = \{(t,x)|T-t\leq\delta, |x-x_0|\leq\varepsilon\} - \{(t,x)|T-t<\delta/2, |x-x_0|<\varepsilon/2\}.$$

Since, by USC, v^+ is bounded on $\{\delta/2 \leq T-t \leq \delta, |x-x_0| \leq \varepsilon/2\}$ we can choose k large enough such that

$$v^+ \leq \psi^k - \delta' \text{ on } \{\delta/2 \leq T-t \leq \delta, |x-x_0| \leq \varepsilon/2\}.$$

Together with (4.5), we obtain

$$v^+ \leq \psi^k - \delta' \text{ on } D.$$

In addition

$$H(t,x,\psi^k_x(t,x),\psi^k_{xx}(t,x)) = H(t,x,\psi_x(t,x),\psi_{xx}(t,x)) \leq C,\quad T-t\leq\delta, |x-x_0|\leq\varepsilon,$$

so

$$-\psi^k_t(t,x) - H(t,x,\psi^k_x(t,x),\psi^k_{xx}(t,x)) \geq k - C > 0,$$

for k even larger, if $T-t\leq\delta, |x-x_0|\leq\varepsilon$. Following the proof of Step 2 in Theorem 3.1, we use again Proposition 4.1 in [2] and the Dini argument to obtain a stochastic sub-solution $v \in \mathcal{V}^+$ such that $v \leq \psi^k - \delta'/2$ on D.

Now let $\eta < \delta'/2 < \varepsilon$ and define

$$v^k = \begin{cases} v \wedge \Big(\psi^k - \eta\Big), & T - t \leq \delta, |x - x_0| \leq \varepsilon, \\ v, \text{ otherwise.} \end{cases}$$

It follows, using the same stopping argument as in the proof of Theorem 3.1, that $v^k \in \mathcal{V}^+$. But we also have that $v^k(T, x_0) = v^+(T, x_0) - \eta < v^+(T, x_0)$, which contradicts the definition of the function v^+.

□

5. Verification by comparison

Before we go ahead, we recall that our analysis rests on the assumption of the existence of stochastic sub and super-solutions. Such assumption may actually be non-trivial to check, especially given the choice of the gauge function ψ (see Remark 5.1 below).

Assumption 5.1. *There is a comparison principle between USC sub-solutions and LSC super-solutions within the class* $|w| \leq C\psi$ *for the PDE*

$$\min[w(x) - g(x), G(T, x, w_x(x), w_{xx}(x))] = 0, \quad \text{on} \quad \mathcal{O}. \tag{5.1}$$

Remark 5.1. *The choice of* ψ *can make a difference whether we have or not a comparison result for* (5.1). *As mentioned, we do not have boundary conditions per-se (this carries over to* (5.1)*), but the information on behavior of solutions near the boundary might, sometimes, be contained in the choice of* ψ. *Therefore, if one wants, for example, to add a constant to* ψ, *having an easier time checking for the existence of stochastic super-solutions or sub-solutions, uniqueness may be lost in* (5.1).

Lemma 5.1. *Let us suppose that Assumption 5.1 and assumptions in both Theorem 3.1 and Theorem 4.1 hold. Then:*

$$v^-(T-, \cdot) = v^+(T, \cdot) = \hat{g}(\cdot), \tag{5.2}$$

where $\hat{g}$ *is the unique continuous viscosity solution of* (5.1). *In addition, both the strong and the weak value functions have well defined limits at* T, *equal to the terminal condition* $\hat{g}$:

$$\lim_{(t<T, x') \to (T, x)} V(t, x') = \lim_{(t<T, x') \to (T, x)} \mathfrak{V}(t, x') = \hat{g}(x), \; x \in \mathcal{O}.$$

Proof. It follows from their definitions that $v^- \leq v^+$. Since v^+ is USC, then

$$v^-(T-, x) = \liminf_{(t<T, x') \to (T, x)} v^-(t, x') \leq \limsup_{(t<T, x') \to (T, x)} v^+(t, x') \leq v^+(T, x). \tag{5.3}$$

Moreover, $v^-(T-, \cdot)$ is a LSC viscosity super-solution of (5.1) as a result of Theorem 3.1, and $v^+(T, \cdot)$ is an USC viscosity sub-solution of the same PDE due to Theorem 4.1. In addition, under the assumptions that both $\mathcal{V}^-$ and $\mathcal{V}^+$ are non-empty, we have the bounds

$$|v^-|, |v^+| \leq C\psi,$$

obtaining therefore similar growth conditions for $v^+(T, \cdot)$ and $v^-(T-, \cdot)$. Thanks to the comparison assumption, it follows that $v^+(T, \cdot) = v^-(T-, \cdot)$ and the common value is the unique continuous viscosity solution of (5.1) that we denote by $\hat{g}$.

In order to prove the second statement, we only need to note that

$$v^- \leq V \leq \mathfrak{V} \leq v^+$$

and plug the equality $v^-(T-, \cdot) = v^+(T, \dot{)} = \hat{g}(\cdot)$ in (5.3).

□

Proposition 5.1. (G upper envelope of g.) *Under Assumption 5.1, the function $\hat{g}$ is the smallest (continuous) function above g which is a viscosity super-solution of*

$$G(T, x, w_x(x), w_{xx}(x)) = 0, \quad \text{on } \mathcal{O}. \tag{5.4}$$

Proof. We know that $\hat{g} \geq g$ and that $\hat{g}$ is a viscosity super-solution of (5.4). Consider now a $w \geq g$ and w is a super-solution of (5.4). Then, w is a super-solution of (5.1). Since $\hat{g}$ is a solution of (5.1) and we have a comparison result, then $\hat{g} \leq w$.

□

Remark 5.2. *When the space of controls is compact, one may take G to be equal to a positive constant. In that case $g = \hat{g}$.*

Definition 5.1. *We say that a comparison principle for* (2.2) *holds if, whenever we have an upper semi-continuous viscosity sub-solution v, and a lower semi-continuous viscosity super-solution w satisfying growth conditions $|v|, |w| \leq C(1+\psi)$ with $v(T, \cdot) \leq w(T, \cdot)$ on $\mathcal{O}$, then $v \leq w$.*

Remark 5.3. *One cannot expect comparison up to time $t = 0$ for semi-continuous viscosity semi-solutions, unless the viscosity property holds in the whole parabolic interior, which includes $t = 0$. This can be seen, for example, from* [6] *and* [8]. *The reader may note that we did prove the viscosity semi-solution property for v^- and v^+ in the parabolic interior.*

Now we are ready to state the main result of this section, which follows as a corollary of Theorems 3.1 and 4.1 and Lemma 5.1.

Theorem 5.1. *Let us assume that a comparison principle for* (2.2) *holds. Moreover, we assume that Assumption 5.1 and assumptions in both Theorem 3.1 and Theorem 4.1 hold. Then, there exists a unique continuous (up to T) viscosity solution $v \in C([0, T] \times \mathcal{O})$ of the PDE* (2.2) *with terminal condition $v(T, \cdot) = \hat{g}(\cdot)$, satisfying the growth condition $|v| \leq C\psi$. Before time T we have:*

$$v(t, x) = v^-(t, x) = v^+(t, x) = V(t, x) = \mathfrak{V}(t, x) \quad (t, x) \in [0, T) \times \mathcal{O}.$$

Proof. Since Assumption 5.1 holds, then $v^-(T-, \cdot) = v^+(T, \cdot) = \hat{g}(x)$. We now define the (still LSC) function

$$w(t, x) = \begin{cases} v^-(t, x), & 0 \leq t < T, x \in \mathcal{O} \\ \hat{g}(x), & t = T, x \in \mathcal{O}. \end{cases}$$

By definition, $w \leq v^+$. At the same time, the function w is a LSC viscosity super-solution and v^+ is a USC viscosity sub-solution of (2.2). Since $v^+(T\cdot) = w(T, \cdot)$ we can use comparison to conclude that $v^+ \leq w$, so

$$v^+ = w \in C([0, T] \times \mathcal{O}).$$

Denoting by $v = w = v^+$, the proof is complete. □

Remark 5.4. *When the controls are unbounded, the value function may display a discontinuity at the terminal time T, as we expect that $v(T-, \cdot) = \hat{g}$ and $v(T, \cdot) = g$. (If $t \neq T$, it follows from the above theorem that the value function is continuous.) The discontinuity was already observed by Krylov in* [15] *on page 252, but the question of what the correct boundary condition should be was left open. For a particular model of super-hedging, an answer was given in* [4]. *The technical arguments to treat such behavior close to the final time horizon were extended to more general models of super-hedging in* [7], [18]. *A summary of such arguments can also be found in* [17] *or in the textbook* [16]. *One of our contributions is to show that this boundary condition holds without relying on the DPP. The proof of the boundary condition comes out as a simple conclusion from the Stochastic Perron method.*

Remark 5.5. *(Fleming-Vermes) As we have mentioned in the introduction, using our notation, Fleming-Vermes* [13] *and* [12] *prove that (with the notation* (4.1)*) we have*

$$V = \mathfrak{v} = \inf\{\textit{classic super-solutions}\},$$

under some technical assumptions (in particular, there is no boundary layer). The proof uses a sophisticated approximation/separation argument, and the probabilistic representation of V, $\mathfrak{v}$.

The program we propose in the present paper can be summarized as

$$\boxed{\text{Theorem 3.1+ Theorem 4.1+Comparison} \rightarrow V = \mathfrak{v} = \text{unique viscosity solution}}$$

However, in the absence of a comparison result for semi-continuous viscosity solutions, little can actually be said about the properties of the value function, following this approach.

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