MATH 705, Seminar in Probability (39948R), Spring 2011.
Class meetings: F 9:30-10:50 am, in KAP 414.

Information on this and linked pages changes frequently.

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Office Hours: MWF 11am-12pm.
Walk-ins and appointments at other time are welcome.

The objective this semester: To discuss the new book ``Brownian Motion'' by P. Morters and Y. Peres (Cambridge University Press, 2010)

The plan: One chapter (about 30 pages) per week. Registered students are expected to make short presentations on the material every week.

The action

January 14: Organizational meeting.

January 21: Sample paths of the Brownian motion.
1. Is there an easy argument showing that W is not Holder of order bigger than 1/2? For non-differentiability, a characteristic function argument seems to work. Can we make it work for the Holder case as well?
2. There are three ways to represent a stationary zero-mean Ornstein-Uhlenbeck (OU) process: by an SODE, by the covariance function by a rescaled Brownian motion

What does rescaling produce in the case of the unstable OU process? What if we formally allow complex numbers in the square roots?
3. The limit of the sum $(W(t_k)-W(t_{k-1}))$ depends on the partition, and we now know how.
4. Is there an easy way to see why, in the modulus of continuity of the Bronwian motion, we get the (natural?) log and the factor of two. There is also the law of iterated log, and it might be interesting to compare the two.
5. What is exchangeability for finite collections of random variables, and is there a connection between exchangeability and tail events for countably many random variables?
6. Is the Brownian filtration continuous for all $t>0$? As the lack of continuity at zero shows, the continuity of the process does not imply continuity of its filtration: you need something extra.
7. What is a Dirichlet space?

January 28: Brownian motion and the strong Markov property.
1. What is a stable subordinator of index α?
2. What other Wald's lemmas are out there? Some might be called identities.
3. What other processes can be embedded into one- and two-dimensional Brownian motions?
4. If $T=\inf\{t: W(t)=1\}$, then $ET=+\infty$, $W(T)=1$ and $EW^2(T)=1$. If $S_n=\min(T,n)$, $n\geq 1$, then $S_n\leq T$ for all n and $EW^2(S)=ES>1$ for sufficiently large n.
5. Non-monotone change of variables can lead to loss of Markov property.

February 4. Brownian motion and partial differential equations.
We can learn a lot by comparing probabilistic and analytic formulations of the same PDE results. For example, compare probabilistic Harnak inequality (equivalence of harmonic measures) and analytic Harnak inequality (comparison of harmonic functions).

February 11. Hausdorff dimension and related topics.
1. If x_n monotonically converges to zero and x_n-x_{n-1} is of order $n^{1/\alpha}$, then can we argue that the Minkowski dimension of the corresponding set is α?
2. Under what conditions will the Minkowski and Hausdorff dimensions coincide?
3. What is the Hausdorff dimension of other Cantor-type sets? For example, if we remove the middle fifth, will it be log 4/log 5?
4. We need connections between the mathematical concepts of α-potential, α-energy, and capacity with the corresponding counterparts in physics (potential energy and capacitance).
5. What is Holder continuity of the Cantor ladder and how can we use the Cantor ladder as a measure in the corresponding theorems for computing the Hausdorff dimension via capacity?
6. Two questions inspired by Frostman's lemma: (a) How to construct the measure? (b) Will every measure μ supported on the set of positive α-Hausdorff measure have the property $\mu(D)\leq C(\mu) |D|^{\alpha}$ for all Borel sets D?

February 18. Brownian motion and random walk, part 1.
1. What is so special about the function $(2t\ln \ln t)^{1/2}$ and what other functions can appear as envelopes of the Brownian motion, both for the whole trajectory and for a sequence of samples?
2. We now know three arcsine laws for the Brownian motion on $[0,1]$: for the

distribution of the time T_0 of last zero, for the distribution of time T_M the max is achieved, and for the distribution of the Lebesgue measure L_+ of the set when the process is positive. Some of the related questions are (a) connections among the three, in particular, whether there is a relation between large/small values of L_+ and T_0; (b) what can be extended to other related processes (for example, the first one works for M-B as well); (c) what are the corresponding results for the (simple, symmetric) random walk; (d) are there any other arcsine laws?
3. What is an example of a non-trivial binary splitting martingale (which is not built out of iid random variables with two values), and what is so good about such martingales?

February 25. Brownian motion and random walk, part 2.
We have two ways to show that Bessel process, if defined as the norm of the standard Brownian motion B in two or more dimensions, satisfies the corresponding SODE. The generator does resemble a part of the Bessel equation, but a more convincing connection between the Bessel process and the Bessel ODE is still missing.
An unrelated observation: M (the running maximum process) is not Markov, even though B, M-B, and 2M-B are all Markov.

March 4. Brownian local time.
We have two characterization of the local time: via the suitable limit of the occupation measure and via the count of under-crossings. Both are useful. Do we have any other characterizations?
There seems to be an easy proof that M is Holder continuous just like B. The Holder continuity of the local time in both time and space seems believable, but the connection between the local time as a function of space and the two-dimensional Brownian motion remains mysterious.

March 11. Conformal invariance, winding number, and Tanaka's formula.
In these circles, conformal mapping in the plane means a bijection defined by an analytic function. The reason could be that the usual definition (that the derivative of the function is not zero) does not narrow down the class of analytic functions, because the Brownian motion on the plane does not hit points, and so the zeros of the derivative can be removed without changing the domain in any meaningful way.
We convinced ourselves that the analytic mapping of a Brownian motion is a time-changed Brownian motion, and if the mapping is conformal (i.e. bijective), then the hitting time of the boundary maps to the hitting time of the boundary. We also discussed an example (using the function f(z)=z^2) where the image hits the boundary later than the pre-image.
The winding number is the argument of the two-dimensional Brownian motion and grows in time as ln(t).

March 18. Potential theory of the Brownian motion.
We discussed the probabilistic proof of the mean-value inequality for sub-harmonic functions, and noticed that if u is harmonic (sub- or super-harmonic), then u(W(t)) is a martingale (respectively, sub- or super-martingale).
There are two ways to write the solution of the equation u_{xx}-\lambda u=f, \lambda >0, in R (and its d-dimensional analogue): probabilistic and by the Fourier transform. The challenge is to see that the formulas give the same result. It also appears that one can take \lambda=0 when d is at least 3.
As far as problem 8.4, some examples of studying the smoothness of probabilistic

solutions are in Krylov's book "Introduction to the theory of diffusion processes", Chapter VI, starting with Theorem VI.1.1.

April 1. No meeting.

April 8. We made some progress on reconciling the two representations of the solution of u_{xx}-\lambda u=f. Most of the time was integration by parts exercises. One question that arose during the discussion was sufficient conditions on a stopping time T and a martingale M to have E M(T)=E M(0).

April 15. We got some understanding of the percolation limit set, and set the goal of writing a Matlab code to generate it in 2d. One of the immediate objectives now is to understand the meanings of capacity and polar set. A general note is that everything done for the Brownian motion can be tried for a more general diffusion, as well as solutions of stochastic partial differential equations.

April 22. A demonstration and discussion of the MatLab program generating a 2d percolation limit set.

April 29. Discussed the cut points of the planar Brownian motion, and the plans for the next semester.

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