Some methods of solving PDEs[1]

Separation of variables

The idea is very simple: when trying to solve a partial differential equation, see if there is a solution of the form $F(x)+G(y)$ or $F(x)G(t)$ or something like this. For example, you can look for solutions of $u_{xx}+u_{yy}=0$ in the form $F(x)+G(y)$ and get answers in the form

$$u(x,y)=ax+b+cy+d.$$

The equation does not have to be linear. For example, if we look for solutions of

$$u_t=uu_{xx}$$

in the form $u(t,x)=F(t)G(x)$, we conclude

$$F'G=FGFG''$$

or

$$F'/F^2=G''=c=1$$

[we can put $c=1$ because we are looking for *some* solution; as an exercise, you can always try $c=2$ or $c=-1$ and see what happens.] Then $F(t)=-t^{-1}$, $G(x)=x^2/2$ work, and you can verify that $u(t,x)=-x^2/(2t)$ satisfies the original equation: $u_t=x^2/(2t^2)=uu_{xx}$.

As another example, you can try solutions of

$$xu_x-yu_y=0$$

in the form $u(x,y)=F(xy)$. Then, by the chain rule, $u_x=yF'(xy)$, $u_y=xF'(xy)$, so that the function $u(x,y)=F(xy)$ satisfies the equation for every continuously differentiable function F. Was it a lucky guess?[2]

The method of characteristics.

The first-order homogeneous linear PDE in n independent variables $x=(x_1,\ldots,x_n)$ can be written in the form

(1)
$$\mathbf{b}(x)\cdot\nabla u(x)=0.$$

The vector function $\mathbf{b}$ can be taken as a right-hand side of an ordinary differential equation:

(2)
$$X'(t)=\mathbf{b}(X(t))$$

Note that if u is a solution of (1) and $X=X(t)$ is a solution of (2), then, by the chain rule, the function $g(t)=u(X(t))$ satisfies

(3)
$$g'(t)=X'(t)\cdot\nabla u(X(t))=\mathbf{b}(X(t))\cdot\nabla u(X(t))=0$$

that is, g is constant. Accordingly, every solution of (2) is called a `characteristic curve` of equation (1), and then (3) implies that every solution of (1) is constant on every characteristic curve. This fact can be used to solve (1) and is especially effective in the case of two independent variables: to solve

(4)
$$a(x,y)u_x+b(x,y)u_y=0$$

we find the general solution of the system

$$x'(t)=a(x(t),y(t)),\quad y'(t)=b(x(t),y(t)),$$

and write the solution as a family of level curves of some function g: $g(x,y)=\text{const}$. Then all solutions of (4) have the form $u(x,y)=F(g(x,y))$ for a continuously differentiable function F.

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[2]No, it was an educated guess, and, if you read through the end, then you will see why

For example, consider

$$xu_x - yu_y = 0.$$

To solve

$$x' = x,\ y' = -y$$

we note that, by dividing the two equations,

$$dy/dx = -y/x, y = c/x, \text{ or } xy = c$$

that is, $u(x, y) = F(xy)$.

Now look at

$$x^2 u_{xx} - y^2 u_{yy} = 0. \tag{5}$$

We see that, for every twice-continuously differentiable function $F = F(t)$, the function $u(x, y) = F(xy)$ is a solution:

$$u_{xx} = y^2 F'', u_{yy} = x^2 F''.$$

A somewhat longer computation shows that, for every twice-continuously differentiable function $G = G(t)$, the function $u(x, y) = G(x/y)$ is also a solution. Let us understand what is happening in this example, in particular, the origin of xy and x/y.

Second-order PDEs in two independent variables: classification and characteristics

Consider the equation

$$A(x, y)u_{xx}(x, y) + 2B(x, y)u_{xy}(x, y) + C(x, y)u_{yy}(x, y) + F(x, y, u, u_x, u_y) = 0; \tag{6}$$

for the purpose of this discussion, we only care about the second-order terms and, in what follow, the remaining terms that are hidden in the function F will often be represented by the dots $\cdots$.

To begin, let us assume that the functions A, B, C are, in fact, constant, that is, real numbers. Then direct computations confirm the following result: there exists a change of variables $(x, y) \to (s, t)$ such that

$$s = c_1 x + c_2 y, \quad t = c_3 x + c_4 y \tag{7}$$

for some numbers c_1, c_2, c_3, c_4 with $c_1 c_4 - c_2 c_3 \neq$ and, depending on the relation among the numbers A, B, C, the new function $v = v(s, t)$ defined from the relation

$$u(x, y) = v(c_1 x + c_2 y, c_3 x + c_4 y)$$

satisfied one of the following three equations:

$$v_{ss} + v_{tt} + \cdots = 0, \text{ if } AC - B^2 > 0;$$
$$v_s = v_{tt} + \cdots, \text{ if } AC - B^2 = 0;$$
$$v_{ss} = v_{tt} + \cdots, \text{ if } AC - B^2 < 0.$$

Now we recall that a generic quadratic equation in two variables

$$ax^2 + 2bxy + cy^2 + c_1 x + c_2 y + c_3 = 0$$

defines[3]

- Ellipse, if $ac - b^2 > 0$;
- Parabola, if $ac - b^2 = 0$;
- Hyperbola, if $ac - b^2 < 0$.

[3] If we stretch the definitions so that $x^2 + y^2 = -1$ is still an ellipse (although empty), $x^2 - y^2 = 0$ is a hyperbola (degenerated into two lines), etc. Alternatively, we interpret the word “generic” so as to exclude the degenerate cases.

As a result, equation (6) is called

- Elliptic, if $AC - B^2 > 0$;
- Parabolic, if $AC - B^2 = 0$;
- Hyperbolic, if $AC - B^2 < 0$.

In this definition, we can now allow A, B, C to be functions of x, y, and then it is possible that the equation is parabolic in one part of the (x, y) plane and hyperbolic in another.

A very special case, such as (5) is when (6) is hyperbolic for all (x, y): in that case, one can (in principle) find the general solution after a suitable change of variables.

Here is how it works. Consider the following ordinary differential equation for the unknown function $y = y(x)$:

$$A(x, y(x))(y'(x))^2 - 2B(x, y(x))y'(x) + C(x, y(x)) = 0. \tag{8}$$

If (6) is always hyperbolic, then $AC - B^2 > 0$ and (8) is equivalent to two first-order ODEs

$$y'(x) = \frac{B \pm \sqrt{B^2 - AC}}{A}. \tag{9}$$

We solve each of them and write the general solutions as level curves of some functions ξ and η:

$$\xi(x, y) = \text{const},\ \ \eta(x, y) = \text{const}.$$

Then REALLY LONG computations show that the function $v = v(\xi, \eta)$ defined by $u(x, y) = v(\xi(x, y), \eta(x, y))$ satisfies

$$v_{\xi\eta} = \cdots$$

and often this equation can be solve by successive integration with respect to ξ and η. In particular, if we are lucky to get

$$v_{\xi\eta} = 0$$

then the solution is

$$v(\xi, \eta) = F(\xi) + G(\eta)$$

for two arbitrary (twice continuously differentiable) functions F and G.

For the first example, consider the wave equation

$$u_{xx} - c^2 u_{yy} = 0. \tag{10}$$

Then, assuming $c > 0$, (9) becomes

$$y'(x) = \pm c$$

or $y \pm cx = \text{const}$, that is,

$$\xi = y - cx,\ \ \eta = y + cx.$$

Then, from $u(x, y) = v(y - cx, y + cx)$, we compute

$$u_x = -cv_\xi + cv_\eta,\ \ u_{xx} = c^2 v_{\xi\xi} - 2c^2 v_{\xi\eta} + c^2 v_{\eta\eta},\ \ u_{yy} = v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta},$$

so that

$$0 = u_{xx} - c^2 u_{yy} = -4c^2 v_{\xi\eta}.$$

that is,

$$v(\xi, \eta) = F(\xi) + G(\eta)$$

for two arbitrary (twice-continuously differentiable) functions. In other words, the general solution of (10) is

$$u(x, y) = F(y - cx) + G(y + cx).$$

You can re-name the variables back to t, x and confirm that the two initial conditions make it possible to identify the functions F and G, leading to D'Alembert's formula.

Now let us de-mystify (5). In this case (8) becomes

$$x^2(y')^2 - y^2 = 0$$

and note that it is the same as

$$(xy' - y)(xy' + y) = 0.$$

Next, with c denoting an arbitrary constant

$$xy' = y$$

implies

$$y = cx,$$

whereas

$$xy' = -y$$

implies

$$y = c/x$$

In other words,

$$\xi = \frac{y}{x}, \ \eta = xy,$$

and then rather long computations show that

$$-4\xi\eta\, v_{\xi\eta} = 0$$

that is, $v(\xi, \eta) = F(\xi) + G(\eta)$. In other words, the general solution of (5) is indeed

$$u(x, y) = F(y/x) + G(xy)$$

for arbitrary (twice continuously differentiable) functions F and G.