Math 606, Summer 2021[1]: Singular Stochastic ODEs; version of June 5, 2021

HOMEWORK PROBLEMS

(1) Let X be standard normal, let $\xi_1, \xi_2, \dots$ be independent exponential random variables with mean 1, and let N be a Poisson random variable with mean $r^2/2$, $r > 0$. Assume that all the random variables are independent. Confirm that $(X+r)^2$ and $X^2 + 2\sum_{k=1}^{N} \xi_k$ have the same distribution. [Compare the moment generating functions.]

(2) Let $f = f(x)$, $x \in \mathbb{R}$, be a function such that $|f(x) - f(y)|^p \le |x-y|^q$ and $0 < p < q$. Show that the function is constant.

(3) Give an example of a Markov process that is not a martingale, and another example of a martingale that is not a Markov process.

(4) Construct an elementary differentiable function $f = f(x)$, $0 < x \le 1$, such that $1 \le f(x) \le 2$ and $\int_0^1 x|f'(x)|\,dx = +\infty$. [The reason is Example 5.13 on page 101 of the book].

(5) Consider the ordinary differential equation

$$a(x)y''(x) + b(x)y'(x) = h(x) \tag{1.1}$$

with $a(x) > 0$ and a known function $h = h(x)$.

(a) Write the general solution of this equation. Assume as much [or as little] as you need about the functions a, b, h.

(b) Let $y = y(x)$ be a solution of (1.1). Find a change of variables $u = p(x)$ so that the function $g = g(u)$ defined by $y(x) = g(p(x))$ satisfies the equation $A(u)g''(u) = H(u)$ and identify the functions A, H.

(6) Confirm that a convex function is continuous and has one-sided derivatives at every point. A useful definition of convexity in this setting is

$$f(y) \le \frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z),\ x < y < z.$$

(7) With sgn $=$ sgn(x) denoting the sign function (with the convention that it is equal to -1 for $x \le 0$), investigate solvability of the ODE $x'(t) = \mathrm{sgn}\big(x(t)\big)$. How is this equation different from $x'(t) = -\mathrm{sgn}\big(x(t)\big)$ (considered in the book)?

(8) A very different example of a singular SODE is

$$dX(t) = -\frac{X(t)}{1-t}\,dt + dW(t),\ 0 < t < 1.$$

Confirm that if $X(0) = 0$, then

$$X(t) = (1-t)\int_0^t \frac{dW(s)}{1-s},$$

so that $X(1-) = 0$ and and $X = X(t)$ is a *Brownian bridge* on $[0,1]$: a zero-mean Gaussian process with $\mathbb{E}X(t)X(s) = \min(t,s) - ts$.

(9) A standard example showing that continuity of the process does not imply continuity (or even right-continuity) of the corresponding filtration is $X(t) = \xi t$, $t \ge 0$, for a (non-degenerate) random variable ξ: $\mathcal{F}_t^X = \sigma(\xi)$, $t > 0$, and $\mathcal{F}_0^X = \{\Omega, \emptyset\}$. Can you construct an example of a continuous process X for which $(\mathcal{F}_t^X)_{t\ge 0}$ is not right-continuous at two points $t_2 > t_1 > 0$? Recall that $\mathcal{F}_t^X = \sigma\big(X(s),\ s \le t\big)$.

(10) Let $X = X(t)$ be a measurable adapted process with $\mathbb{E}|X(t)| < \infty$, $X(0) = 0$, and such that $\mathbb{E}X(\tau) = 0$ for every bounded stopping time τ. Show that X is a martingale. [To show that $\mathbb{E}(X(t)|\mathcal{F}_s) = X(s)$, $t > s$, take $A \in \mathcal{F}_s$ and consider $\tau = sI(A) + tI(A^c)$]

(11) Confirm that, for a continuous martingale M and a fixed x, the corresponding local time process $t \mapsto L^x(t)$ is a sub-martingale. [This follows directly from the Tanaka formula and the fact that $|M|$ is a sub-martingale.] As an extra (super) bonus, think about the process $x \mapsto$

[1]Sergey Lototsky, USC

$L^x(t)$ for fixed t. [In the case of Brownian motion, this question is the subject of a research paper by Edwin Perkins: *Local time is a semimartingale*, Z. Wahrsch. Verw. Gebiete 60 (1982), no. 1, 79–117. The Ray-Knight theorems suggest that it is more interesting to consider the process $x \mapsto L^x(\tau)$ for suitable stopping times τ.]

(12) Confirm that the local time $L = L^x(t)$ of a continuous semi-martingale X satisfies

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I\big(x - \varepsilon \leq X(s) \leq x + \varepsilon\big)\, d\langle X\rangle_s = \frac{L^x(t) + L^{x-}(t)}{2}.$$

How will this result look like for a general semi-martingale?

(13) Confirm that if $X(t) = \int_0^t \operatorname{sgn}\big(X(s)\big)\, dW(s)$, then $\mathcal{F}_t^W = \mathcal{F}_t^{|X|}$, $t \geq 0$. [First note that $W(t) = \int_0^t \operatorname{sgn}\big(X(s)\big)\, dX(s)$. Next, using two definitions of the local time L of X at zero, conclude that, by Tanaka's formula, $|X(t)| = W(t) + L(t)$, whereas, from the previous problem, $L(t)$ is $\mathcal{F}_t^{|X|}$-measurable.]

(14) Consider a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and two stopping times τ, σ.

(a) Confirm that $\tau \wedge \sigma = \min(\tau, \sigma)$ and $\tau \vee \sigma = \max(\tau, \sigma)$ are stopping times and $\mathcal{F}_{\tau\wedge\sigma} = \mathcal{F}_\tau \bigcap \mathcal{F}_\sigma$.

(b) Confirm that the events $\{\tau = \sigma\}$ and $\{\sigma \leq \tau\}$ are $\mathcal{F}_{\tau\wedge\sigma}$-measurable, and the event $\{\sigma < \tau\}$ is $\mathcal{F}_{\tau-}$-measurable.

(c) What is the relation between $\mathcal{F}_{\tau\vee\sigma}$ and $\sigma(\mathcal{F}_\tau \bigcup \mathcal{F}_\sigma)$?

(d) What is the relation between $\mathcal{F}_{\tau+\sigma}$ and $\sigma(\mathcal{F}_\tau \bigcup \mathcal{F}_\sigma)$?

(e) Confirm that if $\tau = T > 0$ is non-random, then $\mathcal{F}_\tau = \mathcal{F}_T$.

(15) Let $A = A(t)$, $t \geq 0$ be a continuous increasing process and define the process $\tau(t) = \min\{s \geq 0 : A(s) = t\}$. Confirm that τ is a continuous increasing process, and, for each $t \geq 0$, the random variable $\tau(t)$ is a stopping time [with respect to the filtration to which A is adapted].

(16) Let T be a positive random variable ($\mathbb{P}(0 < T < \infty) = 1$). Define the process $X = X(t)$ by $X(t) = I(T = t)$. Identify sufficient (and, if possible, necessary) conditions on the distribution of T for each of the following to happen:

(a) The process X has a modification that is identically equal to zero.

(b) The conditions of the Kolmogorov continuity criterion hold.

(c) The process X does not have a modification that is identically zero.

(d) The filtration generated by X is (right-, left-, simply) continuous.

How the answers to (a)–(d) change if T is a stopping time (on a stochastic basis satisfying the usual conditions).

(17) Let $W = W(t)$ be a standard Wiener process and let τ be a stopping time. Confirm that

$$\frac{1}{3}\mathbb{E}\sqrt{\tau} \leq \mathbb{E}\Big(\sup_{t\leq\tau} |W(t)|\Big) \leq 3\mathbb{E}\sqrt{\tau}.$$

[This result extends to any continuous square-integrable martingale: L-Sh, Theory of Martingales, Section 1.9.4, Theorem 5.]

(18) Confirm that a non-negative local martingale is a super-martingale, and it is a martingale if and only if the expected value does not depend on time.

(19) Let $\boldsymbol{W} = \boldsymbol{W}(t)$, $t > 0$, be a standard Brownian motion in $\mathbb{R}^3$ and let $\boldsymbol{a} \in \mathbb{R}^3$ be a fixed non-zero point. With $|\cdot|$ denoting the Euclidean norm, define

$$X(t) = \frac{1}{|\boldsymbol{a} + \boldsymbol{W}(t)|},\ t \geq 0.$$

- Confirm that the process X has the same distribution as the (weak) solution of the equation

$$dY(t) = -Y^2(t)dB(t),\ Y(0) = \frac{1}{|\boldsymbol{a}|}, \tag{1.2}$$

where B is a one-dimensional standard Brownian motion. [Use Itô formula. Keep in mind that the function $F(\boldsymbol{x}) = 1/|\boldsymbol{x}|$ is harmonic in $\mathbb{R}^3$ away from the origin. Use Lévy characterization of the Brownian motion to confirm that $\int_0^t \nabla F\big(\boldsymbol{W}(s)\big)\cdot d\boldsymbol{W}(s)$ is a standard Brownian motion.]

- Confirm that the function $f(t) = \mathbb{E}X(t)$ is monotonically decreasing to zero as $t \to +\infty$ and determine the rate of decay. [Note that $1/X^2(t)$ has non-central chi-square distribution with three degrees of freedom.]
- Investigate a similar problem in $\mathbb{R}^{\mathrm{d}}$ with $\mathrm{d} = 2, 4, 5, \ldots$.

(20) Fill in the details in the following argument. If W is a standard Brownian motion and

$$V(t) = W(t) - \int_0^t \frac{W(u)}{u}\,du,$$

then V is a standard Brownian motion [V is (obviously) Gaussian, and, by direct computation, $\mathbb{E}V(t) = 0$ and $\mathbb{E}|V(t) - V(s)|^2 = |t-s|$], and therefore V is a martingale relative to its own filtration $\mathcal{F}_t^V$, but V is not a martingale relative to $\mathcal{F}_t^W$. Indeed, if $t > s > 0$, then

$$\mathbb{E}\Big(V(t)|\mathcal{F}_s^W\Big) = W(s) - \int_0^s \frac{W(u)}{u}\,du - \int_s^t \frac{W(s)}{u}\,du = V(s) - W(s)\ln(t/s).$$

In other words, $\mathcal{F}_t^V \subsetneqq \mathcal{F}_t^W$ (strict inclusion).

(21) Fill in the details in the following argument. Define the function $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(x) = \begin{cases} (x, 0), & x < 0; \\ (\sin x, 1 - \cos x), & 0 \le x \le 2\pi; \\ (x - 2\pi, 0), & x > 2\pi. \end{cases}$$

The curve $x \mapsto f(x)$ is the x-axis together with the unit circle centered at $(0,1)$. If $W = W(t)$ is a standard Brownian motion, then the process $X(t) = f\big(W(t)+\pi\big)$ is Markov [the inverse f^{-1} exists everywhere except for $(0,0)$, so, with probability one, $W(t) = f^{-1}(X(t)) - \pi$.] On the other hand, the process X is not strong Markov. Indeed, we cannot "restart" X at the stopping time $\tau = \inf\{t > 0 : |W(t)| > \pi\}$ because the behavior of $X(t)$, $t > \tau$, will depend on whether $W(\tau) = \pi$ or $W(\tau) = -\pi$.

(22) Given a measurable function $\sigma = \sigma(x),\ x \in \mathbb{R}$, and a standard Brownian motion W, consider the equation

$$dX(t) = \sigma\big(X(t)\big)\,dW(t). \tag{1.3}$$

Confirm the following.

- If (1.3) has a (weak) solution, then, with probability one, the solution does not explode.
- If $X(0) = a > 0$, $\sigma(x) = 0,\ x \le 0$, and, for every $0 < \varepsilon < M < \infty$,

$$0 < \inf_{\varepsilon < x < M} \sigma(x) \le \sup_{\varepsilon < x < M} \sigma(x) < \infty, \tag{1.4}$$

then equation (1.3) has a unique weak solution. Moreover, condition

$$\int_0^a \frac{x}{\sigma^2(x)}\,dx = +\infty$$

implies X never hits zero in finite time:

$$\mathbb{P}\Big(\inf\{t > 0 : X(t) = 0\} = +\infty\Big) = 1, \tag{1.5}$$

whereas

$$\int_0^a \frac{x}{\sigma^2(x)}\,dx < +\infty$$

implies X hits zero in finite time with probability one:

$$\mathbb{P}\Big(\inf\{t > 0 : X(t) = 0\} < +\infty\Big) = 1. \tag{1.6}$$

If, in the case (1.6), we continue X by zero after hitting zero, then, because the hitting time is a continuous random variable supported on $(0,+\infty)$, we have $\mathbb{P}\big(X(t)=0\big)>0$ for every $t>0$.

- If (1.4) holds and
$$\int_a^{+\infty}\frac{x}{\sigma^2(x)}\,dx<+\infty,$$
then X is a local martingale but not a martingale. [For all of this, and more, see *No Arbitrage Condition for Positive Diffusion Price Processes* by Freddy Delbaen and Hiroshi Shirakawa, Asia-Pacific Financial Markets, volume 9, pages 159–168, 2002.]

(23) The properties of (1.3) imply that, for $r>1$, the weak solution of $dX(t)=X^r(t)dW(t)$, $X(0)=a>0$, never hits zero and is a strict local martingale (a local martingale that is not a martingale). What is the asymptotic, as $t\to+\infty$, of $\mathbb{E}X(t)$?

(24) The geometric Brownian motion $dX(t)=X(t)dW(t)$, with non-random initial condition $X(0)>0$, is an example of (1.3) when the solution is both strictly positive and a martingale: $\mathbb{E}X(t)=X(0)$. The corresponding function $\sigma(x)=x$ is linear. The solution is also strong, in probabilistic sense. Are there any *elementary* non-linear functions σ so that the corresponding solution of (1.3) is strictly positive and a martingale? Can any of the corresponding solutions be strong?

(25) Confirm that, for $x>0$, $a\in\mathbb{R}$, and $\eta\neq 0$, the solutions of equations
$$dX(t)=aX(t)dt+\big(X(t)+\eta I(X(t)=0)\big)dW(t),\ X(0)=x,$$
and
$$dY(t)=aY(t)dt+Y(t)dW(t),\ Y(0)=x,$$
are the same, but the first equation has no solution if $x=0$.

(26) Given the real numbers a,b,c,σ,x, write the solution of the equation
$$dX(t)=\Big(aX(t)+b\Big)dt+\Big(\sigma X(t)+c\Big)dW(t),\ t>0;\ X(0)=x.$$
[Use variation of parameters.]

(27) Consider the equation
$$dX(t)=aX^2(t)\,dt+bX(t)\,dW(t),\ t>0,\quad X(0)=x, \tag{1.7}$$
with non-random $a>0,\ b>0,\ x>0$.
- Apply the Itô formula to $Y(t)=1/X(t)$ to solve the equation.
- For what values of a,b,x do we have $0<\mathbb{P}(\text{no explosion})<1$? [for all $a>0,b>0,x>0$?]

(28) Using your favorite software package,
- generate a few sample paths of the solution of (1.3) with $\sigma(x)=|x|^r$ for $r=\frac{1}{3},\frac{1}{2},\frac{3}{4}$ and $X(0)=1$;
- generate a non-trivial solution of (1.3) with $\sigma(x)=|x|^{1/3}$ and $X(0)=0$.

(29) Using your favorite software package, generate a few sample paths of the solution of the equation
$$dX(t)=\frac{a-1}{2X(t)}I\big(X(t)\neq 0\big)\,dt+dW(t),\ t>0,\ X(0)=x,$$
by mixing and matching different (interesting) values of a and x, such as
$a=-1,0,0.5,1,1.5,2,3$ and $x=0,1$.

(30) Using your favorite software package, generate a sample path $t\mapsto L^x(t)$ of the local time of the standard Brownian motion for $x=0$ and $x=1$.

(31) For what real values of p does the integral $\int_0^1|W(t)|^p\,dt$ converge? [$p>-1$, by Engelbert-Schmidt 0-1 law].

(32) In the spirit of Chapter 4 of the book, identify the type of $\pm\infty$ for the equation $dX(t)=\mu X(t)dt+dW(t)$, $\mu\in\mathbb{R}$.

Basic ideas.

(1) Key number: $48 = 7 \cdot 7 - 1$.
(2) There are more than 48 possibilities for SODEs with one isolated singular point, and a precise count is hardly possible (or useful).
(3) Singular SODEs vs singular SPDEs.
(4) Brownian bridge as a very different singular SODEs.
(5) Wright-Fisher model and its continuum limits.
(6) Strong existence vs weak existence; pathwise uniqueness vs uniqueness in law.
(7) Yamada-Watanabe theorem and its dual (technically, by Engelbert and Cherny).
(8) Martingale vs local martingale vs strict local martingale.
(9) Occupation time vs occupation measure.
(10) Scale function and removal of the drift.
(11) Solvable SODEs (affine, Bernoulli, separable in Stratonovich form).
(12) (Strong) Markov process vs (Strong) Markov family.
(13) Singular point vs boundary point.
(14) Random time change.
(15) Engelbert-Schmidt zero-one law.

Reflective questions for discussions.[2]

(1) Take one homework problem you have worked on this semester that you struggled to understand and solve, and explain how (or if...) the struggle itself was valuable.
(2) What mathematical ideas are you curious to know more about as a result of taking this class? Give one example of a question about the material that you would like to explore further, and explain why you consider this question interesting.
(3) What three theorems did you most enjoy from the course, and why?
(4) Formulate a research question related to the course material that you would like to answer.
(5) Reflect on your overall experience in this class by describing an interesting idea that you learned, why it was interesting, and what it tells you about doing or creating mathematics.
(6) Think of one particular proof [of a result related to the topic of this class] and share your ideas about the ways you think the proof should be improved.

[2]Most are not mine, including the wording. Suggestions for improvement will be part of the discussion.