the following approximate large-sample $100(1-\alpha)\%$ confidence interval for $t(\theta)$:

$$t(\hat{\theta}) \pm z_{\alpha/2}\sqrt{\left[\frac{\partial t(\theta)}{\partial \theta}\right]^2 \Bigg/ nE\left[-\frac{\partial^2 \ln f(Y \mid \theta)}{\partial \theta^2}\right]}$$

$$\approx t(\hat{\theta}) \pm z_{\alpha/2}\sqrt{\left(\left[\frac{\partial t(\theta)}{\partial \theta}\right]^2 \Bigg/ nE\left[-\frac{\partial^2 \ln f(Y \mid \theta)}{\partial \theta^2}\right]\right)\Bigg|_{\theta=\hat{\theta}}}.$$

We illustrate this with the following example.

EXAMPLE **9.18** For random variable with a Bernoulli distribution, $p(y \mid p) = p^y(1-p)^{1-y}$, for $y = 0, 1$. If $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n from this distribution, derive a $100(1-\alpha)\%$ confidence interval for $p(1-p)$, the variance associated with this distribution.

Solution As in Example 9.14, the MLE of the parameter p is given by $\hat{p} = W/n$ where $W = \sum_{i=1}^{n} Y_i$. It follows that the MLE for $t(p) = p(1-p)$ is $\widehat{t(p)} = \hat{p}(1-\hat{p})$.

In this case,

$$t(p) = p(1-p) = p - p^2 \quad \text{and} \quad \frac{\partial t(p)}{\partial p} = 1 - 2p.$$

Also,

$$\begin{aligned}
p(y \mid p) &= p^y(1-p)^{1-y} \\
\ln[p(y \mid p)] &= y(\ln p) + (1-y)\ln(1-p) \\
\frac{\partial \ln[p(y \mid p)]}{\partial p} &= \frac{y}{p} - \frac{1-y}{1-p} \\
\frac{\partial^2 \ln[p(y \mid p)]}{\partial p^2} &= -\frac{y}{p^2} - \frac{1-y}{(1-p)^2} \\
E\left\{-\frac{\partial^2 \ln[p(Y \mid p)]}{\partial p^2}\right\} &= E\left[\frac{Y}{p^2} + \frac{1-Y}{(1-p)^2}\right] \\
&= \frac{p}{p^2} + \frac{1-p}{(1-p)^2} = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}.
\end{aligned}$$

Substituting into the earlier formula for the confidence interval for $t(\theta)$, we obtain

$$\begin{aligned}
t(\hat{p}) &\pm z_{\alpha/2}\sqrt{\left\{\left[\frac{\partial t(p)}{\partial p}\right]^2 \Bigg/ nE\left[-\frac{\partial^2 \ln p(Y \mid p)}{\partial p^2}\right]\right\}\Bigg|_{p=\hat{p}}} \\
&= \hat{p}(1-\hat{p}) \pm z_{\alpha/2}\sqrt{\left\{(1-2p)^2 \Big/ n\left[\frac{1}{p(1-p)}\right]\right\}\Bigg|_{p=\hat{p}}} \\
&= \hat{p}(1-\hat{p}) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})(1-2\hat{p})^2}{n}}
\end{aligned}$$

as the desired confidence interval for $p(1-p)$. ■
