A Summary of the Simple Symmetric Random Walk on the Line.

Main objects.

$X_1, X_2, \ldots :$ iid $\mathbb{P}(X_k = \pm 1) = \dfrac{1}{2}$; $S_0 = 0$, $S_n = \displaystyle\sum_{k=1}^{n} X_k$;

$u_{2n} = \mathbb{P}(S_{2n} = 0) = 2^{-2n}\dbinom{2n}{n}$ (Note : $S_{2n+1} \neq 0$);

$\tau = \min\{k \geq 1 \mid S_k = 0\}$: the time of the first return to zero;

$f_{2n} = \mathbb{P}(\tau = 2n)$;

$L_{2n} = \max\{1 \leq k \leq 2n \mid S_k = 0\}$: the time of the last return to zero on $2n$ steps;

$\pi_{2n} = \#\{k \mid 1 \leq k \leq 2n,\ S_{k-1} > 0 \text{ and/or } S_k > 0\}$.

Main relations among the main objects.

$$\mathbb{P}(S_1 \neq 0, \ldots, S_{2n} \neq 0) = \mathbb{P}(S_{2n} = 0) \equiv u_{2n}; \tag{1}$$

$$\mathbb{P}(\tau > 2n) = u_{2n}; \tag{2}$$

$$f_{2n} = u_{2(n-1)} - u_{2n} \equiv \frac{u_{2(n-1)}}{2n}; \tag{3}$$

$$\mathbb{P}(L_{2n} = 2k) = u_{2k} u_{2(n-k)}; \tag{4}$$

$$\mathbb{P}(\pi_{2n} = 2k) = u_{2k} u_{2(n-k)}. \tag{5}$$

Main results.

(1) THE BALLOT THEOREM: If, in an election with two candidates A and B, the number of votes $N_A > N_B$ and votes are equally likely and independent, then the probability that A was always ahead of B is $(N_A - N_B)/(N_A + N_B)$.

(2) Two ARCSINE LAWS: if α is a random variable with Beta$(1/2, 1/2)$ distribution, then, as $n \to \infty$, both $\frac{L_{2n}}{2n}$ and $\frac{\pi_{2n}}{2n}$ converge in distribution to α.

Note: The Beta(p, q) pdf is

$$\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)};\ B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\ \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, dt.$$

With $p = q = 1/2$, this is

$$\frac{1}{\pi\sqrt{x(1-x)}},\ x \in (0,1). \tag{6}$$

By direct computation, the corresponding cdf is

$$\int_0^x \frac{dt}{\pi\sqrt{t(1-t)}} = \frac{2}{\pi} \sin^{-1}\sqrt{x},\ 0 \leq x \leq 1,$$

whence the `arcsine law`. The function (6) blows up near 0 and 1 and achieves min value at 1/2 meaning that a typical path $\{S_1, S_2, S_3, \ldots\}$ is NOT like a sine wave: it is (much) more likely to stay positive or negative for a long time that to switch frequently between positive and negative values.

Main tool: reflection principle.

STATEMENT. If $k, m \in \mathbb{N} = \{1, 2, \ldots\}$, $a, n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, and $n > a$, then the number of (right, up/down) paths on the integer grid $\mathbb{Z}^2$ from (a, k) to (n, m) that hit zero is the same as the total number of (right, up/down) paths from $(a, -k)$ to (n, m).

TWO MAIN IDEAS OF THE PROOF.

- `Coupling`: merge two paths once they meet;
- `Bijection`: to show that two finite sets have the same number of elements, establish a one-to-one correspondence between the elements of the sets.

The proof of the reflection principle: coupling of the two paths that start as mirror images of each other and meet on the x-axis establishes the bijection.

A possible line of thought: From reflection principle to the ballot theorem, then derive relation (1), from which (2) follows by definition of τ, and then (3) follows from $\mathbb{P}(\tau = 2n) = \mathbb{P}\big(\tau > 2(n-1)\big) - \mathbb{P}\big(\tau > 2n\big)$. The proof of (4) is similar to the proof of the reflection principle; (5) can be proved by induction. The arcsine laws follow by Stirling with $k, n \to \infty$, $k/n \to x$:

$$u_{2n} \sim \frac{1}{\sqrt{\pi n}}, \quad u_{2k} u_{2(n-k)} \sim \frac{1}{\pi \sqrt{k(n-k)}},$$

$$\sum_{2k=\lfloor 2na \rfloor}^{\lfloor 2nb \rfloor} u_{2k} u_{2(n-k)} \sim \frac{1}{\pi n} \sum_{k=\lfloor na \rfloor}^{\lfloor nb \rfloor} \frac{1}{\sqrt{(k/n)(1-(k/n))}} \to \int_a^b \frac{dx}{\pi \sqrt{x(1-x)}}.$$