Stochastic Ordinary Differential Equations: A Summary.

Basic setting:

- $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, stochastic basis with the usual assumptions, that is, $\mathbb{P}$-completeness of $\mathcal{F}_0$ and right continuity of $\mathcal{F}_t$
$$\bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t,\ t \geq 0.$$
- $W = W(t)$, $t \geq 0$, a Wiener process on $\mathbb{F}$, that is, a continuous square-integrable martingale,
$$W(0) = 0, \quad \mathbb{E}\Big(\big(W(t) - W(s)\big)^2 | \mathcal{F}_s\Big) = t - s,\ t \geq s.$$
- Measurable real-valued functions $b = b(x),\ \sigma = \sigma(x),\ x \in \mathbb{R}$, such that
$$\int_0^T \Big(|b\big(f(t)\big)| + |\sigma\big(f(t)\big)|^2\Big)\, dt < \infty \tag{1}$$
for ever $T > 0$ and every $f \in \mathcal{C}([0,T])$, that is, f is continuous on $[0,T]$.

The equation
$$dX = b(X)dt + \sigma(X)dW(t),\ t > 0,$$
understood as the Itô integral equation
$$X(t) = X(0) + \int_0^t b\big(X(s)\big)\, ds + \int_0^t \sigma\big(X(s)\big)\, dW(s),\ t \geq 0, \tag{2}$$
with $X(0) \in \mathcal{F}_0$, that is, X_0 is $\mathcal{F}_0$-measurable.

Basic regularity conditions:
$$\sigma^2 \in \mathcal{C}(\mathbb{R});\ \sigma^2(x) \geq \delta > 0,\ x \in \mathbb{R}; \tag{3}$$
$$|b(x)| + |\sigma(x)| \leq C,\ C > 0,\ x \in \mathbb{R}. \tag{4}$$

Note that (4) implies (1), and (1) is necessary to define the right-hand side of (2).

Two concepts of solution.

Strong solution. Given the stochastic basis and the Wiener process, a `strong solution` $X = X(t)$ of (2) is a continuous $\mathcal{F}_t$-adapted process satisfying (2) with probability one for all $t > 0$ [because X is continuous, the exceptional set does not depend on time]. `Strong uniqueness`, also known as `path-wise uniqueness`, on $[0,T]$ for equation (2) means
$$\mathbb{P}\Big(\sup_{0\leq t\leq T} |X(t) - Y(t)| > 0\Big) = 0$$
for every two solutions $X = X(t)$ and $Y = Y(t)$ satisfying (2) with $X(0) = Y(0)$ and with the same Brownian motion W.

Weak solution, sometimes also called `martingale solution`, is a stochastic basis satisfying the usual conditions, a Brownian motion on this basis, and a continuous $\mathcal{F}_t$-adapted process $X = X(t)$ satisfying (2) with probability one for all $t > 0$. `Weak uniqueness` on $[0,T]$ for equation (2) means that, as a random element with values in $\mathcal{C}([0,T])$, every weak solution has the same distribution.

Local existence of a solution means there is stopping time τ on the underlying stochastic basis such that $\mathbb{P}(\tau > 0) = 1$ and the solution exists on the stochastic interval $\{(\omega, t) : 0 \leq t \leq \tau(\omega)\}$.

General results.

(1) Under the minimal regularity conditions (3), (4), there are only two possibilities: either (2) has a unique strong solution on every stochastic basis with a given Brownian motion, or (2) does not have a unique strong solution regardless of the choice of stochastic basis and Brownian motion; see [4, Section 2.1]. More generally, there are $16 = 2^4$ combinations to consider, namely existence and uniqueness for strong and weak solutions; see [1, Table 1.1]. Of course, many of the combinations do not make sense and some are trivial.

(2) Yamada-Watanabe Theorem: Weak existence and strong uniqueness imply existence and uniqueness of a strong solution [7, Theorem IX.1.7].

(3) Dual Yamada-Watanabe Theorem: Weak uniqueness and strong existence imply existence and uniqueness of a strong solution [A. S. Cherny (2001)].
(4) A strong solution satisfies $\sigma\big(X(s),\ 0 \le s \le t\big) \subseteq \sigma\big(X(0), W(s)\ 0 \le s \le t\big)$, that is, the solution is determined by the initial condition and the Wiener process; a weak solution satisfies $\sigma\big(X(0), W(s)\ 0 \le s \le t\big) \subseteq \sigma\big(X(s),\ 0 \le s \le t\big)$, that is, the initial condition and the Wiener process might not be enough to define the solution; cf. [6, Section 4.4]. In general, it is a separate, and hard, problem to establish equality $\sigma\big(X(0), W(s),\ 0 \le s \le t\big) = \sigma\big(X(s),\ 0 \le s \le t\big)$ of the "input" and "output" sigma-algebras for a strong solution of (2); cf. [5, Section 12.2].
(5) Conditions (4) ensure `non-explosion`: every solution that exists locally can be continued to $[0, T]$ for all non-random $T > 0$. Accordingly, any other condition that ensures non-explosion can be used instead of (4). Two main examples are linear growth and monotonicity:

$$|b(x)| + |\sigma(x)| \le C(1 + |x|),\ C > 0,\ x \in \mathbb{R}; \tag{5}$$

$$xb(x) + \sigma^2(x) \le C(1 + x^2),\ C > 0,\ x \in \mathbb{R}. \tag{6}$$

Linear growth implies monotonicity; taking $b(x) = -x^3$, we see that it is possible to have monotonicity without linear growth. For details and more examples, see [9, Chapter 10].

Basic existence and uniqueness results.

(1) If (3) holds, then equation (2) has a unique weak solution as long as any of conditions for non-explosion, such as (4), (5), or (6), hold (see [8, Theorem 5.6] for the proof under condition (4))
(2) If the functions b and σ are locally Lipschitz continuous, that is, for every $R > 0$ there exists a number L such that

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \le L|x - y|, \tag{7}$$

for all x, y with $|x| \le R$, $|y| \le R$, then equation (2) has a unique strong solution as long as any of conditions for non-explosion, such as (4), (5), or (6), hold (see [3, Theorem V.1.1] for the proof under condition (6)). It possible to relax (7) in a way similar to (6): for every $R > 0$ there exists a number L such that

$$(x - y)\big(b(x) - b(y)\big) + |\sigma(x) - \sigma(y)|^2 \le L|x - y|^2 \tag{8}$$

for all x, y with $|x| \le R$, $|y| \le R$; cf. [3, Theorem V.1.1].

More sophisticated existence and uniqueness results.

(1) If $\sigma^2 \ge \delta > 0$, then equation (2) has a unique weak solution as long as any of conditions for non-explosion, such as (4), (5), or (6), hold (see [2] for the proof under condition (4)). In other words, no continuity of σ^2 is required.
(2) If (3) holds and $|\sigma(x) - \sigma(y)| \le L\sqrt{|x - y|}$ for some $L > 0$ and all $x, y \in \mathbb{R}$, then equation (2) has a unique strong solution as long as any of conditions for non-explosion, such as (4), (5), or (6), hold (see [10] for the proof under condition (4)).

For a comprehensive collection of examples, see [1, Section 1.3].

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