A Summary of Renewal Theory.

Main objects.

$$
\begin{aligned}
& X_1, X_2, \ldots: \ \text{ iid } \ (\text{inter} - \text{renewal intervals})\\
& \mathbb{P}(X_k \geq 0) = 1,\ \mathbb{E}X_k = \mu > 0;\ S_0 = X_0 \geq 0 \text{ indep. of all other } \mathrm{X_k},\\
& S_n = S_0 + \sum_{k=1}^{n} X_k;\\
& F = F(x)\ (\text{cdf of } X_1),\ F_n = F^{*n}(x)\ (\text{cdf of } S_n - S_0),\ G = G(x)\ (\text{cdf of } X_0);\\
& N_t = \sum_{n=1}^{\infty} 1(S_n \leq t) \equiv \sup\{n \mid S_n \leq t\} = \max\{n \mid S_n \leq t\} \ \text{ because } \ \mathbb{P}(N_t < \infty) = 1;\\
& U(A) = \sum_{n=0}^{\infty} \mathbb{P}(S_n \in A)\ (\text{renewal measure}),\ U(t) = U((0,t]) \equiv \mathbb{E}N_t\ (\text{renewal function}).
\end{aligned}
$$

If $X_0 = 0$, then $N_0 = 0$. If, in addition, X_1 is exponential with mean $1/\lambda$ (or `rate parameter` λ), then $N = N_t$ is Poisson process with intensity λ and $U(t) = \lambda t$.

Some references consider a different process N*, namely,* $\bar{N}_t = \inf\{n \mid S_n > t\}, t \geq 0$, so that $\bar{N}_t = N_t + 1$, and, in particular, $\bar{N}_0 = 1$. A potential advantage of $\bar{N}$ is that there is no need to worry whether $X_0 = 0$. The main disadvantage is that $\bar{N}$ seem less common, partly because $\bar{N}$ is NOT a Poisson process when $X_0 = 0$ and $X_k,\ k \geq 1$, are exponential.

Definitions. The RENEWAL PROCESS is either the sequence $\{S_n, n \geq 0\}$ or the (continuous time) process $N = N_t,\ t \geq 0$. The implicit assumption is that $X_0 = 0$; otherwise, the term DELAYED RENEWAL PROCESS is used. Finally, a RENEWAL REWARD PROCESS is

$$
W_t = \sum_{k=1}^{N_t} R_k \equiv \sum_{k=1}^{\infty} R_k 1(S_k \leq t) \ \ (\text{because } \{\omega \mid S_n \leq t\} = \{N_t \geq n\},\ \ \{\omega \mid S_n > t\} = \{N_t < n\}),
$$

where the random variables R_k are iid, but R_k and X_k can be dependent and each R_k can be negative (i.e. the term "rewards" includes costs too).

Basic results. Assume that $X_0 = 0$. Then

$$
\begin{aligned}
& \text{LLN}: \lim_{t\to+\infty} \frac{N_t}{t} = \frac{1}{\mu}; \qquad \lim_{t\to+\infty} \frac{W_t}{t} = \frac{\mathbb{E}R_1}{\mu} \text{ with probability one;}\\
& \text{CLT}: \lim_{t\to+\infty} \frac{N_t - (t/\mu)}{\sqrt{t/\mu^3}} = \mathcal{N}(0,\sigma^2) \text{ in distribution } \ (\text{if } \mathrm{Var}(X_1) = \sigma^2 < \infty).
\end{aligned}
$$

Note:

(1) Recall that

$$
(H * F)(x) = \int_{-\infty}^{+\infty} H(x-y)\, dF(y); \quad F^{*n}(x) = (F^{*(n-1)} * F)(x).
$$

(2) Proofs of LLN, CLT and many other results in renewal theory use a squeeze theorem-type argument; some results, such as LLN, hold even with $\mu = +\infty$.

(3) Some results require X_1 to be non-arithmetic: $|\mathbb{E}e^{\sqrt{-1}tX_1}| < 1, t \neq 0$.

Basic Renewal Theorems.

$$
\begin{aligned}
& \lim_{t\to+\infty} \frac{U(t)}{t} = \frac{1}{\mu},\\
& \lim_{t\to+\infty} U([t, t+s]) = \frac{s}{\mu} \quad (\text{Blackwell, non} - \text{arithmetic } X_1).
\end{aligned}
$$

Renewal Equations.

$$\textbf{Basic}: \quad U(t) = F(t) + (U*F)(t) \equiv F(t) + \int_0^t U(t-s)\,dF(s)$$

$$\text{Solution}: \quad U(t) = \sum\nolimits_{k=1}^{\infty} F_n(t);$$

$$\textbf{General}: \quad H(t) = h(t) + (H*F)(t), \quad \text{known } h$$

$$\text{Solution}: \quad H(t) = h(t) + (h*U)(t) : U*F = U-F, H*F = H*U = H-h;$$

$$\textbf{General renewal theorem}: \quad \lim_{t\to\infty} H(t) = \frac{1}{\mu}\int_0^{+\infty} h(t)\,dt.$$

Note: for the process $\bar{N}_t$ with $\bar{N}_0 = 1$, the corresponding renewal function $\bar{U}(t) = U(t) + 1$ satisfies $\bar{U} = 1 + \bar{U} * F$.

Inspection/waiting paradox can be expressed in various ways:

(1) For $t > 0$, the random variable $X_{N_t+1} = S_{N_t+1} - S_{N_t}$ (length of the inter-renewal interval containing t) `stochastically dominates` the random variable X_1 (typical waiting time), that is, $\mathbb{P}(X_{N_t+1} > x) \geq \mathbb{P}(X_1 > x)$;
(2) the *mean total lifetime* of the light bulb currently in use is larger than the *mean lifetime of a typical light bulb*;
(3) in the limit $t \to \infty$, the distribution of X_{N_t+1} converges to the `size-biased` distribution of X_1, with cdf $F^\star(x) = \int_0^x (y/\mu)dF(y),\ x > 0$.

Main example: optimal replacement strategy.

THE QUESTION. An object has a random life time τ; the replacement cost is A if the object has not failed yet and $B > A$ if the object failed. Assume that the object is a part of the system that will be operating for a long time. What is the optimal schedule for replacing the object to minimize the long-term running cost?

THE SETTING. This is renewal reward process: renewal is replacement, reward is the cost. Let x be the time of replacement and let F_τ be the cdf of τ. Then the expected service time of the object is

$$\mu(x) = \mathbb{E}(\tau|\tau < x)\mathbb{P}(\tau < x) + x\mathbb{P}(\tau \geq x) = \int_0^x y\,dF_\tau(y) + x\big(1 - F_\tau(x)\big) = \int_0^x \big(1 - F_\tau(y)\big)\,dy.$$

and the expected replacement cost is

$$r(x) = B\mathbb{P}(\tau < x) + A\mathbb{P}(\tau > x) = A + (B-A)F_\tau(x).$$

By the LLN, the cost over the time period $[0,t]$, for large t, is approximately $tC(x) = t\,r(x)/\mu(x)$, so that the optimal $x^* = \arg\min C(x)$.

UNIFORM CASE: τ is uniform on $[0,T]$. Then

$$\mu(x) = x - x^2/(2T), \quad r(x) = A + (B-A)x/T, \ C(x) = \frac{2AT + 2x(B-A)}{2Tx - x^2},$$

and, with $\alpha = A/(B-A)$,

$$x^* = \Big(\sqrt{\alpha^2 + 2\alpha} - \alpha\Big)T < T.$$

Wikipedia example: $T = 2$, $\alpha = 1/12$, $x^* = 2/3$.

Note that $x^* \to T$ as $B \to A$ so that $\alpha \to +\infty$, and, for small α, $x^* \approx (\sqrt{2\alpha} - \alpha)T$.

EXPONENTIAL CASE: if $F_\tau(y) = 1 - e^{-\lambda y}$, then

$$\lambda\mu(x) = 1 - e^{-\lambda x} = F_\tau(x), \quad \frac{C(x)}{\lambda} = B - A + \frac{A}{1 - e^{-\lambda x}},$$

so the optimal strategy is $x^* = +\infty$: "do not fix it if it is not broken" (!?!)

General Reference: *Renewal Theory* by D. R. Cox (b. 1924); first published in 1962, about 150 pages.