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On the Probability that k Positive Integers are Relatively Prime

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Let $P_k(n)$ denote the probability that k positive integers, chosen at random from $\{1, 2,..., n\}$, are relatively prime. It is shown that $P_k(n) = 1/\zeta(k) + O(1/n)$ if $k \geqslant 3$ and $P_2(n) = 1/\zeta(2) + O(\log n/n)$, where ζ denotes the Riemann ζ-function. Hence for $k \geqslant 2$, $\lim_{n\to\infty} P_k(n) = 1/\zeta(k)$. The same problem is studied using probability distributions on the positive integers other than the uniform distribution on $\{1, 2,..., n\}$ as was used above. The following result, with examples, is given: Let f be a probability density function, defined on the cartesian product of the set of positive integers with itself k times, which has the following property: if $(m_1,..., m_k) = d$, then $f(m_1,..., m_k) = g(d)f(m_1/d,..., m_k/d)$ for some function g defined on the positive integers. Then the probability that a k-tuple of positive integers chosen from this distribution is relatively prime is $1/\sum_{d=1}^{\infty} g(d)$.

1. The following notation will be used:

$Z_k(t)$ = number of k-tuples of positive integers less than or equal to t which are relatively prime.

$P_k(n)$ = probability that k positive integers, chosen at random from $\{1, 2,..., n\}$ are relatively prime.

ζ = Riemann ζ-function.

LEMMA. *For $t \geqslant 1$, $Z_k(t) = t^k/\zeta(k) + O(t^{k-1})$ if $k \geqslant 3$ and $Z_2(t) = t^2/\zeta(k) + O(t \log t)$.*

Proof. Note that

$$Z_k(t) = \sum_{\substack{(m_1,\ldots,m_k)=1 \\ 1\leqslant m_i\leqslant t}} 1 \tag{1}$$

and

$$[t]^k = \sum_{\substack{1 \leqslant m_i \leqslant t \\ (i=1,\ldots,k)}} = \sum_{1 \leqslant d \leqslant t} \sum_{\substack{(m_1,\ldots,m_k)=d \\ 1 \leqslant m_i \leqslant t}} 1 \tag{2}$$

Now $(m_1, \ldots, m_k) = d$ if and only if $(m_1/d, \ldots, m_k/d) = 1$. Hence there is a 1–1 correspondence between k-tuples $\langle m_1, \ldots, m_k \rangle$ with $(m_1, \ldots, m_k) = d$ and $1 \leqslant m_i \leqslant t$ and k-tuples $\langle m_1', \ldots, m_k' \rangle$ with $(m_1', \ldots, m_k') = 1$ and $1 \leqslant m_i' \leqslant t/d$. However there are exactly $Z_k(t/d)$ of the latter. Hence from (1) and (2) we obtain

$$[t]^k = \sum_{1 \leqslant d \leqslant t} Z_k(t/d) \tag{3}$$

Applying one of the Moebius inversion formulas [2; p. 104] to (3) yields

$$Z_k(t) = \sum_{1 \leqslant d \leqslant t} \mu(d)[t/d]^k,$$

$$= \sum_{1 \leqslant d \leqslant t} \mu(d)[t/d + O(1)]^k.$$

Hence it is easily seen that

$$Z_k(t) = t^k \sum_{1 \leqslant d \leqslant t} \mu(d)/d^k + t^{k-1} O\Big(\sum_{1 \leqslant d \leqslant t} \mu(d)/d^{k-1}\Big) + \cdots$$

$$+ \, t\, O\Big(\sum_{1 \leqslant d \leqslant t} \mu(d)/d\Big) + O\Big(\sum_{1 \leqslant d \leqslant t} 1\Big). \tag{4}$$

We consider the terms in this equation separately. We have

$$\sum_{1 \leqslant d \leqslant t} \mu(d)/d^k = \sum_{d=1}^{\infty} \mu(d)/d^k - \sum_{d=[t]+1}^{\infty} \mu(d)/d^k.$$

Now it is well known that $\sum_{d=1}^{\infty} \mu(d)/d^k = 1/\zeta(k)$. Also

$$\Big| \sum_{d=[t]+1}^{\infty} \mu(d)/d^k \Big| < \sum_{d=[t]+1}^{\infty} 1/d^k < \int_{[t]}^{\infty} dx/x^k = O(1/t^{k-1}).$$

Hence the first term of (4) is $t^k/\zeta(k) + O(t)$. To estimate the other terms we observe

$$\sum_{1 \leqslant d \leqslant t} \mu(d)/d^i = O(1) \qquad \text{if} \quad i > 1,$$

$$\Big| \sum_{1 \leqslant d \leqslant t} \mu(d)/d \Big| \leqslant \sum_{1 \leqslant d \leqslant t} 1/d = \log t + \gamma + O(1/t) (\gamma \text{ is Euler's constant}),$$

and

$$\sum_{1 \leqslant d \leqslant t} 1 = O(t).$$

From these observations the result is easily seen to follow.

The proof given above is modeled after the derivation of the formula for the summatory function of the Euler ϕ-function given in [2].

THEOREM 1. *$P_k(n) = 1/\zeta(k) + O(1/n)$ if $k \geqslant 3$ and $P_2(n) = 1/\zeta(2) + O(\log n/n)$. Hence $\lim_{n\to\infty} P_k(n) = 1/\zeta(k)$.*

Proof. This follows immediately from the lemma together with the simple observation $P_k(n) = Z_k(n)/n^k$.

The statement $\lim_{n\to\infty} P_k(n) = 1/\zeta(k)$ may roughly be interpreted as saying that if k positive integers are chosen at random, the probability that they are relatively prime is $1/\zeta(k)$. This, however, is not precise since there is no uniform distribution on the positive integers. What we have considered above is the uniform distribution on $\{1, 2,..., n\}$ and then taken the limit as $n \to \infty$. In the following section we consider the same type of problem using other probability distributions on the positive integers.

2. In the following, N will denote the set of positive integers and f will denote a joint probability density function defined on the cartesian product of N with itself k times.

DEFINITION. We will say f satisfies condition A if there exists a function g defined on N such that whenever $(m_1 ,..., m_k) = d$,

$$f(m_1 ,..., m_k) = g(d) f(m_1/d,..., m_k/d).$$

THEOREM 2. *If f satisfies condition A, then the probability P_k, that a a k-tuple of positive integers chosen from this distribution is relatively prime, is $1/\sum_{d=1}^{\infty} g(d)$.*

Proof. Clearly $P_k = \sum_{(m_1,...,m_k)=1} f(m_1 ,..., m_k)$. Now

$$1 = \sum_{d=1}^{\infty} \sum_{(m_1,...,m_k)=d} f(m_1 ,..., m_k) = \sum_{d=1}^{\infty} \sum_{(m_1,...,m_k)=d} g(d) f(m_1/d,..., m_k/d)$$

$$= \sum_{d=1}^{\infty} g(d) \sum_{(m_1',...,m_k')=1} f(m_1',..., m_k') = \sum_{d=1}^{\infty} g(d) P_k .$$

Hence $P_k = 1/\sum_{d=1}^{\infty} g(d)$.

EXAMPLE 1. For $s > 2$ define f_s by $f_s(m_1, m_2) = c/(m_1 + m_2)^s$ where $c = [\zeta(s-1) - \zeta(s)]^{-1}$. To show f_s is a probability density function we observe that the number of ways n can be written as a sum of two positive integers is $n - 1$. Hence

$$\sum_{m_2=1}^{\infty} \sum_{m_1=1}^{\infty} c/(m_1 + m_2)^s = c \sum_{n=2}^{\infty} (n-1)/n^s = c \sum_{n=2}^{\infty} 1/n^{s-1} - c \sum_{n=2}^{\infty} 1/n^s$$
$$= c[\zeta(s-1) - \zeta(s)] = 1.$$

Now f_s satisfies condition A with $g(d) = 1/d^s$.

Hence, by Theorem 2, the probability that a pair of positive integers chosen from this distribution is relatively prime is $1/\zeta(s)$.

EXAMPLE 2. For $s > 1$ define f_s by

$$f_s(m_1, ..., m_k) = \zeta^{-k}(s)(m_1 \cdot m_2 \cdots m_k)^{-s}.$$

It is easy to verify that f_s is a probability density function and that f_s satisfies condition A with $g_s(d) = 1/d^{ks}$. If we let $q_k(s)$ denote the probability that a k-tuple of positive integers chosen from this distribution is relatively prime, then Theorem 2 tells us that $q_k(s) = 1/\zeta(ks)$.

Notice that in Example 1 the underlying probability measure cannot be factored as a product measure, while in Example 2 the underlying probability measure is a product measure (i.e., in Example 2 we have independence while in Example 1 we do not).

3. Comparing the result in Section 1 with the result in Example 2 of Section 2 we observe

$$\lim_{n\to\infty} P_k(n) = \lim_{s\to 1+} q_k(s) = 1/\zeta(k).$$

This does not appear to be coincidental. Let P_s denote the probability distribution on N with density function $f_s(m) = m^{-s}/\zeta(s)$. Golomb [1] discusses the "limit distribution" P_1 obtained by taking suitable limits as $s \to 1$ in P_s. He shows that the distributions P_s can be used to approximate the nonexistant uniform distribution on N in much the same way that the uniform distribution on $\{1, 2, ..., n\}$ can be used to approximate the same thing. With this in mind it seems plausible that the result $\lim_{n\to\infty} P_k(n) = 1/\zeta(k)$, obtained in Section 1, could be obtained from the results in Section 2 and the above observations. At present the author is not able to do this rigorously.

References

1. S. W. Golomb, A Class of Probability Distributions on the Integers, *J. Number Theory* **2** (1970), 189–192.
2. H. Rademacher, "Lectures on Elementary Number Theory," Blaisdell Publishing Co., New York, 1967.