Math 705 AGZ Random Matrices Book Ch.4 - Some Generalities

The general points of the chapter:

- A framework for the derivation of the joint distributions of eigenvalues in the matrix ensembles.
- Derivation of such joint distributions for some classical ensembles: GOE/ GUE/ GSE, Laguerre, Jacobi, unitary ensembles.
- "Determinantal" point processes; eigenvalues of GUE are such. Derivation of a CLT for the number of eigenvalues in an interval; some "ergodic consequences."
- Time-dependant random matrices, entries replaced by Brownian Motion. Allows Ito integration. CLTs, large deviations.
- Concentration inequalities and their applications to random matrices.
- Tridiagonal model of RM, "beta ensemble."

Joint Distributions of eigenvalues

Proposition 1. *For every nonnegative Borel-measurable function ϕ on $\mathcal{H}_n(\mathbb{F})$ s.t. $\phi(X)$ depends only on the eigenvalues of X, we have*

$$\int \phi d\rho_{\mathcal{H}_n(\mathbb{F})} = \frac{\rho[U_n(\mathbb{F})]}{(\rho[U_l(\mathbb{F})])^n n!} \int_{\mathbb{R}^n} \phi(x)|\Delta(x)|^\beta \Pi_{i=1}^n dx_i,$$

where for every $x = (x_1, ..., x_n) \in \mathbb{R}^n$ we write $\phi(x) = \phi(X)$ for any $X \in \mathcal{H}_n(\mathbb{F})$ with eigenvalues $x_1, ..., x_n$.

A couple of more such results follow in the book.

Next, the *coarea formula*: Fix a smooth map $f : M \to N$ from an n-manifold to a k-manifold, with derivative at a point $p \in M$ denoted $\mathbb{T}_p(f) : \mathbb{T}_p(M) \to \mathbb{T}_{f(p)}(N)$. Let $M_{crit}, M_{reg}, N_{crit}$, and N_{reg} be the sets of critical (regular) points (values) of f. (See Definition F.3 and Proposition F.10 in the book for the terminology.) For $q \in N$ s.t. $M_{reg} \cap f^{-1}(q)$ is nonempty, we equip the latter with the volume measure $\rho_{M_{reg}\cap f^{-1}(q)}$. Put $\rho_\emptyset = 0$ for convenience. Also let $J(\mathbb{T}_p(f))$ denote the generalized determinant of $\mathbb{T}_p(f)$. (See Definition F.17 in the book).

Theorem 1. *(Coarea formula) With notation and setting as above, let ϕ be any nonnegative Borel-measurable function on M. Then:*

(i) the function $p \to J(\mathbb{T}_p(f))$ on M is Borel-measurable;

(ii) the function $q \to \int \phi(p) d\rho_{M_{reg}\cap f^{-1}(q)}(p)$ on N is Borel-measurable;

(iii) the integral formula

$$\int \phi(p) J(\mathbb{T}_p(f)) d\rho_M(p) = \int \left(\phi(p) d\rho_{M_{reg}\cap f^{-1}(q)}(p)\right) d\rho_N(q)$$

holds.

The authors say the above is a kind of a Fubini's Theorem.

Lemma 1. *S^{n-1} is a manifold and for every $x \in S^{n-1}$ we have $\mathbb{T}_x(S^{n-1}) = \{X \in \mathbb{R}^n : x.X = 0\}$.*

Proposition 2. *With notation as above, we have*

$$\rho\left[S^{n-1}\right] = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

Lemma 2. *Let $M \subset Mat_{n\times k}(\mathbb{F})$ be a manifold. Fix $g \in GL_n(\mathbb{F})$. Let $f = (p \to gp) : M \to gM = \{gp \in Mat_{n\times k}(\mathbb{F}) : p \in M\}$. Then:*
(i) gM is a manifold and f is a diffeomorphism;
(ii) for every $p \in M$ and $X \in \mathbb{T}_p(M)$ we have $\mathbb{T}_p(f)(X) = gX$;
(iii) if $g \in U_n(\mathbb{F})$, then f is an isometry (and hence measure-preserving).

Lemma 3. *With p, q, n positive integers so that $p + q = n$, and $D = diag(I_p, 0_q)$, the collection $Flag_n(D, \mathbb{F})$ is a manifold of dimension βpq. (Definitions of "Flag" and so on are in the book.)*

Theorem 2. *(Weyl) Let (G, H, M, Λ) be a Weyl quadruple. Then for every Borel-measurable nonnegative G-conjugation-invariant function ϕ on M, we have*

$$\int \phi d\rho_M = \frac{\rho[G]}{\rho[H]} \int \phi(\lambda)\sqrt{det\Theta_\lambda}d\rho_\Lambda(\lambda).$$

Here we use the following definition for *Weyl quadruple*.

Definition 1. *A Weyl quadruple (G, H, M, Λ) consists of four manifolds G, H, M, and Λ with common ambient space $Mat_n(\mathbb{F})$ satisfying the following conditions:*
(I) (a) G is a closed subgroup of $U_n(\mathbb{F})$,
(b) H is a closed subgroup of G, and
(c) $dimG - dimH = dimM - dim\Lambda$.
(II) (a) $M = \{g\lambda g^{-1} : g \in G, \lambda \in \Lambda\}$,
(b) $\Lambda = \{h\lambda h^{-1} : h \in H, \lambda \in \Lambda\}$,
(c) for every $\lambda \in \Lambda$ the set $\{h\lambda h^{-1} : h \in H\}$ is finite, and
(d) for all $\lambda, \mu \in \Lambda$ we have $\lambda^*\mu = \mu\lambda^*$.
(III) There exists $\Lambda' \subset \Lambda$ such that
(a) Λ' is open in Λ,
(b) $\rho_\Lambda(\Lambda\ \Lambda') = 0$, and
(c) for every $\lambda \in \Lambda'$ we have $H = \{g \in G : g\lambda g^{-1} \in \Lambda\}$.
We say that a subset $\Lambda' \subset \Lambda$ for which $(IIIa, b, c)$ hold is generic.

Application of the Weyl Theorem begin at page 209 of the book.

1 Determinantal point processes

Some definitions: Let Λ be a locally compact Polish space, equipped with a (necessarily σ-finite) positive Radon measure μ on its Borel σ-algebra (recall that a positive measure is Radon if $\mu(K) < \infty$ for each compact set K.) Next let $\mathcal{M}(\Lambda)$ denote the space of σ-finite Radon measures on Λ, and let $\mathcal{M}_+(\Lambda)$ denote the subset of $\mathcal{M}(\Lambda)$ consisting of positive measures.

Definition 2. *(a) A point process is a random, integer-valued $\mathcal{X} \in \mathcal{M}_+(\Lambda)$. (By random we mean that for any Borel $B \subset \Lambda$, $\mathcal{X}(B)$ is an integer-valued random variable.)*

(b) A point process $\mathcal{X}$ is simple if

$$P(\exists x \in \Lambda : \mathcal{X}(\{x\}) > 1) = 0.$$

Lemma 4. *A v-distributed random element x of $\mathbb{X}$ can be associated with a point process $\mathcal{X}$ via the formula $\mathcal{X}(B) = |\boldsymbol{x}_B|$ for all Borel $B \subset \Lambda$. If $v(\mathbb{X}^{\neq}) = 1$, then $\mathcal{X}$ is a simple point process.*

Here $\mathbb{X}$ denotes the space of locally finite configurations in Λ, and $\mathbb{X}^{\neq}$ is the space of locally finite configurations with no repetitions. Or more precisely, for $x_i \in \Lambda$, $i \in I$ an interval of positive integers (beginning at 1 if nonempty), with I finite or countable, let $[x_i]$ denote the equivalence class of all sequences $\{x_{\pi(i)}\}_{i \in I}$, where π runs over all permutations (finite or countable) of I. Then we set

$$\begin{aligned}\mathcal{X} = \mathcal{X}(\Lambda) = \{\mathbf{x} = [x_i]_{i=1}^{\kappa},\ &\text{where } x_i \in \Lambda, \kappa \leq \infty,\ \text{and} \\ &|\mathbf{x}_K| := \sharp\{i : x_i \in K\} < \infty \text{ for all compact } K \subset \Lambda\}\end{aligned}$$

and

$$\mathbb{X}^{\neq} = \{\mathbf{x} \in \mathcal{X} : x_i \neq x_j \text{ for } i \neq j\}.$$

We give $\mathbb{X}$ and $\mathbb{X}^{\neq}$ the σ-algebra $\sigma_{\mathcal{X}}$ generated by the cylinder sets $C_n^B = \{\mathbf{x} \in \mathcal{X} : |\mathbf{x}_B| = n\}$, with B Borel with compact closure and n a nonnegative integer.

Now let us introduce one more definition:

Definition 3. *Let $\mathcal{X}$ be a simple point process. Assume locally integrable functions $\rho_k : \Lambda^k \to [0, \infty), k \geq 1$, exist such that for any mutually disjoint family of subsets $D_1, ..., D_k$ of Λ,*

$$E_v\left[\Pi_{i=1}^k \mathcal{X}(D_i)\right] = \int_{\Pi_{i=1}^k D_i} \rho_k(x_1, ..., x_k) d\mu(x_1)...d\mu(x_k).$$

Then the functions ρ_k are called the joint intensities (or correlation functions) of the point process $\mathcal{X}$ with respect to μ.

The authors remark that by Lebesgue's Theorem, for μ^k almost every $(x_1, ..., x_k)$,

$$\lim_{\epsilon \to 0} \frac{P(\mathcal{X}(B(x_i, \epsilon)) = 1, i = 1, ..., k)}{\Pi_{i=1}^k \mu(B(x_i, \epsilon))} = \rho_k(x_1, ..., x_k).$$

Next,

Lemma 5. *Let $\mathcal{X}$ be a simple point process with intensities ρ_k.*

(a) For any Borel set $B \subset \Lambda^k$ with compact closure

$$E_v\left(|x^{\wedge k} \cap B|\right) = \int_B \rho_k(x_1, ..., x_k) d\mu(x_1)...d\mu(x_k).$$

(b) If D_i, $i = 1, ..., r$ are mutually disjoint subsets of Λ contained in a compact set K, and if $\{k_i\}_{i=1}^r$ is a collection of positive integers such that $\sum_{i=1}^r k_i = k$, then

$$E_v\left[\Pi_{i=1}^r \binom{\mathcal{X}(D_i)}{k_i} k_i!\right] = \int_{\Pi D_i^{\times k_i}} \rho_k(x_1, ..., x_k)\mu(dx_1)...\mu(dx_k).$$

We continue with "determinantal processes."

Definition 4. *A simple point process $\mathcal{X}$ is said to be a determinantal point process with kernel K (in short: determinantal process) if its joint intensities ρ_k exist and are given by*

$$\rho_k(x_1, ..., x_k) = det_{i,j=1}^k(K(x_i, x_j)).$$

Also,

Definition 5. *An integral operator $\mathcal{K} : L^2(\mu) \to L^2(\mu)$ with kernel K given by*

$$\mathcal{K}(f)(x) = \int K(x, y)f(y)d\mu(y), \quad f \in L^2(\mu)$$

is admissible (with admissible kernel K) if $\mathcal{K}$ is self-adjoint, nonnegative and locally trace-class, that is, with the operator $\mathcal{K}_D = \boldsymbol{1}_D\mathcal{K}\boldsymbol{1}_D$ having kernel $K_D(x, y) = \boldsymbol{1}_D(x)K(x, y)\boldsymbol{1}_D(y)$, the operators $\mathcal{K}$ and $\mathcal{K}_D$ satisfy:

$$< g, \mathcal{K}(f) >_{L^2(\mu)} = < \mathcal{K}(g), f >_{L^2(\mu)}, \quad f, g \in L^2(\mu),$$

$$< f, \mathcal{K}(f) >_{L^2(\mu)} \geq 0, \quad f \in L^2(\mu),$$

For all compact sets $D \subset \Lambda$, the eigenvalues $(\lambda_i^D)_{i\geq 0}(\in \mathbb{R}^+)$ of $\mathcal{K}_D$ satisfy $\sum \lambda_i^D < \infty$.

We say that $\mathcal{K}$ is *locally admissible* (with locally admissible kernel K) if the two identities above hold with $\mathcal{K}_D$ replacing $\mathcal{K}$.

Lemma 6. *Suppose $K : \Lambda \times \Lambda \to \mathbb{C}$ is a continuous, Hermitian and positive definite function, that is $\sum_{i=1}^n z_i^* z_j K(x_i, x_j) \geq 0$ for any $n, x_1, ..., x_n \in \Lambda$ and $z_1, ..., z_n \in \mathbb{C}$. Then $\mathcal{K}$ is locally admissible.*

Etc... The authors make further definitions of increasing complexity, and use them to derive CLT results.

2 Brownian motion and random matrices

Definition 6. *Let* $(B_{i,j}, \tilde{B}_{i,j}, 1 \leq i, \leq j \leq N)$ *be a collection of iid real valued standard Brownian motions. The symmetric (resp. Hermitian) Brownian motion, denoted* $H^{N,\beta} \in \mathcal{H}_N^\beta, \beta = 1, 2$ *is the random process with entries* $\{H_{i,j}^{N,\beta}(t), t \geq 0, i \leq j\}$ *equal to*

$$H_{k,l}^{N,\beta} = \frac{1}{\sqrt{\beta N}}(B_{k,l} + i(\beta - 1)\tilde{B}_{k,l}), \text{ if } k < l,$$

$$H_{k,l}^{N,\beta} = \frac{\sqrt{2}}{\sqrt{\beta N}} B_{l,l} \text{ if } k = l.$$

Let $(W_1, ..., W_N)$ be a N-dimensional Brownian motion in a probability space (Ω, P) equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$. Let Δ_N denote the open simplex

$$\Delta_N = \{(x_i)_{1 \leq i \leq N} \in \mathbb{R}^N : x_1 < x_2 < ... < x_{N-1} < x_N\},$$

with closure $\overline{\Delta_N}$. With $\beta \in \{1, 2\}$, let $X^{N,\beta}(0) \in \mathcal{H}_N^\beta$ be a matrix with (real) eigenvalues $(\lambda_1^N(0), ..., \lambda_N^N(0)) \in \overline{\Delta_N}$. For $t \geq 0$, let $\lambda^N(t) = (\lambda_1^N(t), ..., \lambda_N^N(t)) \in \overline{\Delta_N}$ denote the ordered collection of (real) eigenvalues of

$$X^{N,\beta}(t) = X^{N,\beta}(0) + H^{N,\beta}(t),$$

with $H^{N,\beta}$ as in the definition above. An important observation partly due to Dyson is that the process $(\lambda^N(t))_{t \geq 0}$ is a vector of semi-martingales, whose evolution is described by a stochastic differential system.

Theorem 3. *Let* $(X^{N,\beta}(t))_{t \geq 0}$ *be as above, with eigenvalues* $(\lambda^N(t))_{t \geq 0}$ *and* $\lambda^N(t) \in \overline{\Delta_N}$ *for all* $t \geq 0$*. Then, the processes* $(\lambda^N(t))_{t \geq 0}$ *are semi-martingales. Their joint law is the unique distribution on* $C(\mathbb{R}^+, \mathbb{R}^N)$ *so that*

$$P(\forall t > 0, (\lambda_1^N(t), ..., \lambda_N^N(t)) \in \Delta_N) = 1,$$

which is a weak solution to the system

$$d\lambda_i^N(t) = \frac{\sqrt{2}}{\sqrt{\beta N}} dW_i(t) + \frac{1}{N} \sum_{j: j \neq i} \frac{1}{\lambda_i^N(t) - \lambda_j^N(t)} dt, \quad i = 1, ..., N$$

with initial condition $\lambda^N(0)$.