

Convergence of Random Variables¹

The setting: probability space $(\Omega, \mathcal{F}, \mathbb{P})$, random variables $\xi_n = \xi_n(\omega)$, $n \geq 1$, and $\xi = \xi(\omega)$, $\omega \in \Omega$.

Definitions.

- (1) **Uniform convergence:** $\xi_n \rightrightarrows \xi$ if $\forall \varepsilon > 0 \exists N = N(\varepsilon) \forall n > N \forall \omega \in \Omega : |\xi_n(\omega) - \xi(\omega)| < \varepsilon$.
- (2) **Point-wise convergence:** $\xi_n \xrightarrow{p.w.} \xi$ if $\xi_n(\omega) \rightarrow \xi(\omega)$ for all $\omega \in \Omega$, that is,
 $\forall \varepsilon > 0 \forall \omega \in \Omega \exists N = N(\varepsilon, \omega) \forall n > N : |\xi_n(\omega) - \xi(\omega)| < \varepsilon$.
- (3) **Complete convergence:** $\xi_n \xrightarrow{c.c.} \xi$ if $\sum_{n=1}^{\infty} \mathbb{P}(|\xi_n - \xi| > \varepsilon) < +\infty$ for every $\varepsilon > 0$.
- (4) **Almost sure convergence:** $\xi_n \xrightarrow{a.s.} \xi$ if $\mathbb{P}\left(\omega : \lim_{n \rightarrow \infty} \xi_n(\omega) = \xi(\omega)\right) = 1$. Also known as convergence with probability one.
- (5) **Convergence in L_p , $p > 0$:** $\xi_n \xrightarrow{L_p} \xi$ if $\lim_{n \rightarrow \infty} \mathbb{E}|\xi_n - \xi|^p = 0$.
- (6) **Convergence in probability:** $\xi_n \xrightarrow{\mathbb{P}} \xi$ if $\lim_{n \rightarrow \infty} \mathbb{P}(|\xi_n - \xi| > \varepsilon) = 0$ for every $\varepsilon > 0$.
- (7) **Convergence in distribution:** $\xi_n \xrightarrow{d} \xi$ if $\lim_{n \rightarrow \infty} \varphi_{\xi_n}(t) = \varphi_{\xi}(t)$ for every $t \in \mathbb{R}$; $\varphi_{\xi}(t) = \mathbb{E}e^{it\xi}$ is the characteristic function of ξ .
- (8) **Uniform integrability:** the sequence $\{\xi_n, n \geq 1\}$ is uniformly integrable (UI) if

$$\lim_{a \rightarrow +\infty} \sup_{n \geq 1} \mathbb{E}\left(|\xi_n| 1(|\xi_n| > a)\right) = 0.$$

Each of the following is a sufficient condition for UI:

- $|\xi_n| \leq \eta$ for all n and $\mathbb{E}\eta < +\infty$;
- $\mathbb{E} \sup_n |\xi_n| < \infty$;
- $\sup_n \mathbb{E}g(|\xi_n|) < \infty$ for a non-negative increasing function $g = g(x)$, $x \geq 0$, such that $\lim_{x \rightarrow +\infty} \frac{g(x)}{x} = +\infty$. For example, $\sup_n \mathbb{E}|\xi_n|^r < \infty$ for some $r > 1$.

General implications:

$$\xrightarrow{p.v.} \Leftarrow \rightrightarrows \Rightarrow \xrightarrow{c.c.} \Rightarrow \xrightarrow{a.s.} \Rightarrow \xrightarrow{\mathbb{P}} \Rightarrow \xrightarrow{d}; \quad \xrightarrow{L_p} \Rightarrow \xrightarrow{\mathbb{P}}.$$

The (non-trivial) reasons. $\rightrightarrows \Rightarrow \xrightarrow{c.c.}$ because \rightrightarrows implies $\mathbb{P}(|\xi_n - \xi| > \varepsilon) = 0$ for all sufficiently large n ;
 $\xrightarrow{c.c.} \Rightarrow \xrightarrow{a.s.}$ by the first Borel-Cantelli; $\xrightarrow{a.s.} \Rightarrow \xrightarrow{\mathbb{P}}$ because $\xrightarrow{a.s.}$ means $\lim_{n \rightarrow \infty} \mathbb{P}(\sup_{k \geq n} |\xi_k - \xi| > \varepsilon) = 0$;
 $\xrightarrow{\mathbb{P}} \Rightarrow \xrightarrow{d}$ by the Dominate Convergence Theorem; $\xrightarrow{L_p} \Rightarrow \xrightarrow{\mathbb{P}}$ by Markov's inequality.

Basic counterexamples, with $\Omega = [0, 1]$ and \mathbb{P} the Lebesgue measure,

- (1) **the Growing Pulse** $\xi_n(x) = n^\alpha 1(0 < x < n^{-\beta})$ or $\xi_n(x) = n^\alpha 1(0 \leq x < n^{-\beta})$ [if DO NOT want ξ_n to converge to zero point-wise], for suitable $\alpha > 0, \beta > 0$, shows that
 - $\xrightarrow{p.v.}$ or $\xrightarrow{a.s.} \not\Rightarrow \xrightarrow{L_p}$ or $\xrightarrow{c.c.}$
 - $\xrightarrow{c.c.} \not\Rightarrow \rightrightarrows$ or $\xrightarrow{p.w.}$
- (2) **the Typewriter Sequence** $\xi_{n,k} = 1(k2^{-n} < x < (k+1)2^{-n})$, $k = 0, \dots, 2^n - 1$, $n = 0, 1, 2, \dots$, shows that

$$\xrightarrow{L_p}, \quad 0 < p < \infty \not\Rightarrow \xrightarrow{a.s.}$$

Special implications

- (1) $\xrightarrow{\mathbb{P}} \Rightarrow \xrightarrow{a.s.}$ if the sequence $\{\xi_n, n \geq 1\}$ is monotone;
- (2) $\xrightarrow{d} \Rightarrow \xrightarrow{\mathbb{P}}$ if the limit ξ is non-random: there exists a real number c such that $\mathbb{P}(\xi = c) = 1$;
- (3) $\xrightarrow{d} \Rightarrow \xrightarrow{a.s.}$ for sums of independent random variables [using Cauchy criterion and zero-one law];

¹Sergey Lototsky, USC

- (4) $\xrightarrow{\mathbb{P}} \Rightarrow \xrightarrow{L_p}$ if the sequence $\{|\xi_n|^p, n \geq 1\}$ is UI.
- (5) $\xrightarrow{a.s.} \Rightarrow \xrightarrow{c.c.}$ if the random variables ξ_n are independent AND the limit ξ is non-random [by the second Borel-Cantelli; if the limit ξ is random, then the random variables $\xi_n - \xi$ might be dependent and the second Borel-Cantelli will not apply];
- (6) If ξ_n are iid and $\mathbb{E}|\xi_n| < \infty$, then $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \xi_k = \mathbb{E}\xi_1$ both with probability one and in L_1 .
- (7) If $\xi_n \xrightarrow{\mathbb{P}} \xi$, then $\xi_{n_k} \xrightarrow{c.c.} \xi$ along a suitable sub-sequence;
- (8) If $\xi_n \xrightarrow{\mathbb{P}} \xi$ and, for some $p > 0$, $\lim_{n \rightarrow \infty} \mathbb{E}|\xi_n|^p = \mathbb{E}|\xi|^p$, then $\{|\xi_n|^p, n \geq 1\}$ is UI and $\xi_n \xrightarrow{L_p} \xi$.

Working with limits

- (1) If $M = a.s.$ or $M = \mathbb{P}$, and $\xi_n \xrightarrow{M} \xi$, $\eta_n \xrightarrow{M} \eta$, then $\xi_n \pm \eta_n \xrightarrow{M} \xi \pm \eta$, $\xi_n \eta_n \xrightarrow{M} \xi \eta$, and, if $\eta \neq 0$, then also $\xi_n/\eta_n \xrightarrow{M} \xi/\eta$.
- (2) If $\xi_n \xrightarrow{M} \xi$, with $M = a.s.$ or \mathbb{P} or d , and $f = f(x)$, $x \in \mathbb{R}$ is a continuous function, then $f(\xi_n) \xrightarrow{M} f(\xi)$. The result is obvious when $M = a.s.$, easy when $M = \mathbb{P}$, and non-trivial [known as the **Mann-Wald Theorem**] when $M = d$.
- (3) **Slutsky's Theorem:** if $\xi_n \xrightarrow{d} \xi$ and $\eta_n \xrightarrow{d} b \in \mathbb{R}$ non-random, then $\xi_n \pm \eta_n \xrightarrow{d} \xi \pm b$, $\xi_n \eta_n \xrightarrow{d} b\xi$, and, if $b \neq 0$, then also $\xi_n/\eta_n \xrightarrow{d} \xi/b$.

Further Facts

- $\xrightarrow{p.w.}$ corresponds to a topology, but the topology is not metrizable;
- $\xrightarrow{\mathbb{P}}$ is metrizable: the function $\rho(\xi, \eta) = \mathbb{E} \frac{|\xi - \eta|}{1 + |\xi - \eta|}$ defines a metric on the space of random variables, and $\xi_n \xrightarrow{\mathbb{P}} \xi$ if and only if $\lim_{n \rightarrow \infty} \rho(\xi_n, \xi) = 0$;
- \xrightarrow{d} is metrizable using the **Lévy-Prokhorov metric**;
- $\xrightarrow{a.s.}$ is not metrizable and does not correspond to any topology.