# Summary[1] of Quaternions.

**Basic definitions.**
The set:

$$\mathbb{H} = \{q = a + b\mathfrak{i} + c\mathfrak{j} + d\mathfrak{k} : a, b, c, d \in \mathbb{R}\}.$$

Multiplication rules,

$$\mathfrak{i}^2 = \mathfrak{j}^2 = \mathfrak{k}^2 = \mathfrak{i}\,\mathfrak{j}\,\mathfrak{k} = -1, \tag{1.1}$$

were (literally) carved in stone [into the stone of Brougham Bridge in Dublin, Ireland] on Monday, October 16, 1843, by Sir William Rowan Hamilton (1805–1865).[2] In particular,

$$\mathfrak{i}\,\mathfrak{j} = -\mathfrak{j}\,\mathfrak{i} = \mathfrak{k}. \tag{1.2}$$

Alternatively, $\mathfrak{i} \to \mathfrak{j} \to \mathfrak{k} \to \mathfrak{i}$. The resulting multiplication on $\mathbb{H}$,

$$\begin{aligned}(a + b\mathfrak{i} + c\mathfrak{j} + d\mathfrak{k})(x + y\mathfrak{i} + z\mathfrak{j} + w\mathfrak{k}) &= (ax - by - cz - dw) + (ay + bx + cw - dz)\mathfrak{i}\\ &+ (az + cx + dy - bw)\mathfrak{j} + (aw + xd + bz - cy)\mathfrak{k}\end{aligned} \tag{1.3}$$

is associative but non-commutative, and turns $\mathbb{H}$ into a *division algebra*.
For $q \in \mathbb{H}$, define

$$q^* = a - b\mathfrak{i} - c\mathfrak{j} - d\mathfrak{k},\ \mathfrak{N}(q) = qq^* = a^2 + b^2 + c^2 + d^2 = q^*q,\ q^{-1} = \frac{q^*}{\mathfrak{N}(q)} \ (\text{if } \mathfrak{N}(q) > 0). \tag{1.4}$$

A versor is a quaternion $q$ with $\mathfrak{N}(q) = 1$; it is used to describe rotations in $\mathbb{R}^3$.
A major *algebraic miracle* (which, of course, follows from (1.2) and (1.4)) is that

$$(q_1q_2)^* = q_2^*q_1^* \tag{1.5}$$

and then

$$\mathfrak{N}(q_1q_2) = q_1q_2q_2^*q_1^* = \mathfrak{N}(q_1)\mathfrak{N}(q_2). \tag{1.6}$$

**Equivalent characterizations** of $\mathbb{H}$:

- $\mathbb{H} \cong \mathbb{R}^4$ [four-dimensional Euclidean space], with $\mathfrak{N}(q) = \|q\|^2$, the Euclidean norm of the corresponding vector, and a (rather ugly) multiplication rule

$$(a, b, c, d)(x, y, z, w) = (ax-by-cz-dw, ay+bx+cw-dz, az+cx+dy-bw, aw+xd+bz-cy); \tag{1.7}$$

  the rule, of course, is the same as (1.3).
- $\mathbb{H} \cong \mathbb{C}^2$ [a pair of complex numbers], by writing $q = (a+b\mathfrak{i})+(c+d\mathfrak{i})\mathfrak{j}$ and using the relation (1.1) for the imaginary units $\mathfrak{i}$ and $\mathfrak{j}$.
- $\mathbb{H} \cong \mathbb{R} \times \mathbb{R}^3$ [a scalar and a vector] by writing $q = a + \vec{u}$, $\vec{u} = b\hat{\imath} + c\hat{\jmath} + d\hat{\kappa}$ and re-writing (1.3) as

$$(a + \vec{u})(x + \vec{v}) = ax - \vec{u}\cdot\vec{v} + \big(a\vec{v} + x\vec{u} + \vec{u}\times\vec{v}\big); \tag{1.8}$$

  it is then natural to call $a$ the real part of $q$ and $\vec{u}$, the vector part. With this interpretation, a versor (unit quaternion)

$$\hat{q} = \cos(\theta/2) + \hat{u}\sin(\theta/2),$$

  with a unit vector $\hat{u} \in \mathbb{R}^3$, represents rotation around the axis in the direction of $\hat{u}$ by the angle $\theta$:

$$\mathbb{R}^3 \ni \vec{v} \mapsto \hat{q}\,\vec{v}\,\hat{q}^{-1},$$

  where $\vec{v}$ on the right-hand side is interpreted as a quaternion with zero real part; by direct computation, the real part of the resulting product will be zero.

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[1]Sergey Lototsky, USC; version of May 4, 2023.
[2]The vectors in $\mathbb{R}^3$ and operations on them first appeared in 1880s, in the works of J. Gibbs and O. Heaviside.

**Algebra background.**

The starting point is a GROUP, a set with a binary operation $(a,b) \mapsto a \circ b$ having a unit and an inverse. Next, we have a RING, a set with two binary operations, "addition" $(a,b) \mapsto a+b$ and "multiplication" $(a,b) \mapsto a \cdot b$; it is an Abelian (commutative) group for $+$; "multiplication" is associative and has the distributive property with "addition", but does not have to be commutative, invertible, or even have a unit. A FIELD is a ring where both operations lead to Abelian groups. A LINEAR SPACE over a field is a collection of objects that allow linear combinations with coefficients in the field; a similar construction with a ring instead of a field is called a MODULE. Finally, an ALGEBRA $\mathcal{A}$ is a linear space over a field with a BI-LINEAR FORM, that is, a bi-linear operation $\mathcal{A} \times \mathcal{A} \ni (a,b) \mapsto [a,b] \in \mathcal{A}$. In a DIVISION ALGEBRA, this bi-linear operation is, in some sense, invertible (and this can be called "multiplication"): for every $a, b$, with $b \neq 0$, there are unique $x, y$ such that $a = [b,x] = [y,b]$.

Here are some facts.

(1) A *finite-dimensional division algebra over the field of real numbers* can only be of dimension 1,2,4, or 8 (as proved in 1958, independently, by Michel Kervaire and John Milnor).

(2) A *commutative and associative* finite-dimensional division algebra over the field of real numbers is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

(3) A *non-commutative but associative* finite-dimensional division algebra over the field of real numbers is isomorphic $\mathbb{H}$.

(4) The main eight-dimensional version, known as OCTONIONS, has multiplication non-commutative and non-associative, but still *alternative*, that is, the "multiplication" operation satisfies $[X,[X,Y]] = [[X,X],Y], [[X,Y],Y] = [X,[Y,Y]]$. Then, by analogy with (1.8), one concludes that a *reasonable* extension of the cross product[3] is only possible in seven dimensions.

(5) A finite-dimensional *commutative* division algebra over the field of real numbers is either 1- or 2-dimensional (as proved in 1940 by Heinz Hopf, of the Hopf algebras fame and an advisor of Michel Kervaire).

(6) A finite-dimensional commutative division algebra over the field of real numbers DOES NOT have to be isomorphic to $\mathbb{C}$. An example is the complex numbers with "multiplication" defined by

$$[(a+b\mathfrak{i}),(x+y\mathfrak{i})] = (a-b\mathfrak{i})(x-y\mathfrak{i}) \equiv (a+b\mathfrak{i})^*(x+y\mathfrak{i})^* \equiv \big((a+b\mathfrak{i})(x+y\mathfrak{i})\big)^* = (ax-by)-(ay+bx)\mathfrak{i}.$$

This operation is commutative but not associative and does not have a unit.

[3]note that cross product is NOT alternative