Instructor: **MATH 407** March 19, 2021
Prof. Arratia **Poisson random variable and Poisson process**

Motivation Thanks to Matlab project C, you are already somewhat familiar with the Bernoulli process, and the first ten arrivals in that process, especially for a tiny value of p, such as $p = .0000001$. The Poisson process is the limit of the Bernoulli process, when $p \to 0$.

Poisson random variables. Let $\lambda > 0$. X is said to have the Poisson distribution *with parameter* λ if, for $k = 0, 1, 2, \ldots$

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}. \tag{1}$$

For $k = 0, 1, 2$ (1) simplifies: $\mathbb{P}(X = 0) = e^{-\lambda}, \mathbb{P}(X = 1) = \lambda e^{-\lambda}, \mathbb{P}(X = 2) = \lambda^2 e^{-\lambda}/2$. Thanks to theTaylor series for the exponential function, $\exp(z) = \sum_{k \geq 0} z^k/k!$ (true for any complex number $z \in \mathbb{C}$,) the values in (1) sum to 1, and X is a proper nonnegative integer valued random variable. For example, under (1), $\mathbb{P}(X > 2) = 1 - \mathbb{P}(X = 0, 1, \text{ or } 2) = 1 - e^{-\lambda}(1 + \lambda + \lambda^2/2)$.

Facts to memorize: Under (1), $\lambda = \mathbb{E}\, X = \operatorname{Var} X$.

Baby exercise: suppose Z is Poisson, with $\mathbb{E}\, Z = 3.2$. Simplify: $\mathbb{P}(Z = 1)/\mathbb{P}(Z = 0) =$ _______, $\mathbb{P}(Z = 2)/\mathbb{P}(Z = 1) =$ _______, $\mathbb{P}(Z = 5)/\mathbb{P}(Z = 4) =$ _______.

Relation to Binomial distributions. Suppose $p_n \to 0$ with $n\, p_n \to \lambda \in (0, \infty)$. Suppose that the distribution of X_n is Binomial(n, p_n) and the distribution of X is Poisson(λ). (This includes the more restrictive case, $p_n = \lambda/n$ for fixed λ, and all $n \geq \lambda$.) Then X_n converges in distribution, meaning that for each fixed k, as $n \to \infty$, $\binom{n}{k} p_n^k (1 - p_n)^{n-k} \to e^{-\lambda} \frac{\lambda^k}{k!}$. For 407, you do not need to know the proof. But you should be able to easily see that $\mathbb{E}\, X_n = n\, p_n \to \lambda = \mathbb{E}\, X$, and $\operatorname{Var} X_n = n\, p_n(1 - p_n) \to \lambda = \operatorname{Var} X$.

An easy way to know whether you should use Binomial(n, p) or Poisson(λ): ask whether can get your hands on n. For example: suppose a cookie dough averages 40 chocolate chips per pound, and X is the number of chips in a small batch of cookies weighing 1 ounce. The exercise is: $\mathbb{P}(X = 0) =$ _______, $\mathbb{P}(X = 3) =$ _______, $\mathbb{P}(X \geq 2) =$ _______. There is no "n" in the story, and we should fit a Poisson distribution with $\lambda = \mathbb{E}\, X = 40/16 = 2.5$.

Poisson process. To get free of *one-dimensional* thinking, we start with a three-dimensional example: meteor strikes from 2019 to 2029, in the United States. We get rid of Alaska and Hawaii, and further simplify the map so that our country is a rectangle, 3000 miles from west to east and 1000 miles from north to south. Suppose the rate of strikes is 1.21 per million square miles per year. (For the overall area of the U.S., the rate is 3.63 per year, and for our 10 year span, we expect 36.3 strikes. Say 'the west' is the western 1/3 of our area, and the rest is the other 2/3. Using the full 10 year span, let X be the number of strikes on the west, (call its mean μ), let Y be the number of strikes on the rest, (call its mean ν, and let $Z := X + Y$ be the number of strikes on the entire country. Write λ for the mean of Z. The appropriate model is: X is Poisson with $\mu = \mathbb{E}\, X = 12.1$, Y is Poisson with $\nu = \mathbb{E}\, Y = 24.2$, X and Y are independent, and Z is Poisson with $\lambda = \mathbb{E}\, Z = 36.3$. [There is an underlying theorem, call it the *Poisson superposition theorem*: X, Y independent Poisson distributed implies that $X + Y$ is also Poisson distributed.] You should try to imagine two alternate ways that the meteors will arrive: a) each square mile, independently of all other square miles, attracts meteor strikes at the rate .00000121 per year; b) God decides to have Z meteors strike the U.S. over these 10 years, according to (1) with Z in the role of X and 36.3 as the value for λ. Having found a value for Z, for example 25, independently for each of the Z meteors, she picks a random time and place to land it! Both constructions are valid; note that each meteor has a $p = 1/3$ chance to land in the west. From b), it *should be plausible* that, conditional on the event $\{Z = 25\}$, the distribution of X is Binomial$(n = 25, p = 1/3)$. Again, there is a theorem, and it is a small extension of the Poisson superposition theorem: if X, Y are independent Poisson, and $Z = X + Y$, then with $p = \mathbb{E}\, X/(\mathbb{E}\, X + \mathbb{E}\, Y)$, for any $0 \le k \le n$,

$$\mathbb{P}(X = k \mid Z = n) = \binom{n}{k} p^k (1-p)^{n-k}. \tag{2}$$

You should be able to imagine further divisions: say California is 1/6 of one million squre miles, hence 1/18 of the entire country. Given that 45 meteors strike the U.S. in the next decade, the number to hit California will be Binomial$(n = 45, p = 1/18)$, and the conditional expectation is $np = 2.5$. Without conditioning, the distribution is Poisson, with mean $36.3/18 = 2.0166....$, slightly more than 2. This is over an entire decade, and the rate per year, for California, is $r = 36.3/180 = .20166....$

The rate r Poisson process, whose full name is: the time homogeneous, one-dimensional Poisson process with rate r. After reading the paragraph below, go back and thing about the *standard* case, which uses $r = 1$. (I present the general case, even though I always end up rescaling time, so the that I only need to deal with the standard case, because if I showed you the standard case first, and then asked you to generalize to allow a rate r, it would be much more confusing.) Our notation will be similar to what we used for the Bernoulli process, except that S_n, for the Binomial distributed number of heads after n tosses (of a p-coin) will be replaced by N_t, for the number of arrivals *by time* t, which is like the number of meteors landing on the interval $(0, t]$.

Notations.
For $t > 0$, N_t is the number of arrivals by time t. We define $N_0 = 0$.
For $k = 1, 2, \ldots$, T_k is the time of the kth arrival. We define $T_0 = 0$.

The interarrival times are $W_1, W_2, \ldots$, so that

$$T_k = W_1 + W_2 + \cdots + W_k,$$

and $W_1, W_2, \ldots$ turn out to be mutually independent, and identically distributed (i.i.d.). The relation tying the counts to the arrival times is an identity of events:

$$\{\omega : N_t < k\} = \{\omega : T_k > t\}. \tag{3}$$

The relation (3) is described as *duality*.

Distributions

N_t is Poisson, with $\lambda := \mathbb{E}\, N_t = rt$.
$W \equiv W_1$ has distribution called *exponential* with *rate* parameter r: for $t \ge 0$,

$$\mathbb{P}(W > t) = \mathbb{P}(N_t = 0) = e^{-rt}.$$

In the above, we started with strict inequality, to respect the duality relation (3), but by continuity, we also have $\mathbb{P}(W = t) = 0$ and $\mathbb{P}(W \geq t) = e^{-rt}$. From the integration-by-parts upper tail formula, $\mathbb{E}\,W = \int_{t\geq 0} \mathbb{P}(W > t)\; dt = \int_{t\geq 0} e^{-rt}\; dt = 1/r$.

All the above distributional facts, and the i.i.d. claim about $W_1, W_2, \ldots$, are proved by taking the limit of a sequence of Bernoulli processes, using $p = p_n = r/n$, and scaling time by a factor of n. Your Matlab project 1b was meant to illustrate this limit, using $n = 1,000,000$, and you should imagine $r = 1$, but if your computer performed 1.2 million seed attempts per second, then, in units of seconds, and after scaling seeds T_i by a million, to get 'running times' around a a second, and using the name T_i again for these rescaled times, you would be approximating the Poisson process with rate $r = 1.2$.

In the meteor strike example, we discussed two ways to imagine constructing the Poisson process; these were labeled a) and b). Now, for the one-dimensional rate r Poisson process, we have a third construction, c): start with i.i.d. exponentially distributed, $W_1, W_2, \ldots$, with $\mathbb{E}\,W_i = 1/r$. Let $T_k = W_1 + \cdots + W_k$, and define N_t by (3), so that N_t is the number of k such that $T_k \in (0, k]$.

It is a sad fact of life that "W is exponential with parameter 2" is not unambiguous; the parameter might be intended as the rate, so that $r = 2$ and $\mathbb{E}\,W = 1/2$, or the parameter might be intended as the mean, so that $\mathbb{E}\,W = 2$ and the rate is $r = 1/2$. To avoid this ambiguity, you should always specify mean, or rate, when giving the parameter for an exponential distribution. The same holds for a more general family, called Gamma distributions, which include the exponential distributions.

Exercises. Calls arrive at a switchboard at an average rate of .2 per second; that is, one every 5 seconds on average. Assume the appropriate Poisson process.

a1) $\mathbb{P}($ wait at least 4 seconds for the first call to arrive$) =$ _______

a) $\mathbb{P}($ wait at least 4 seconds, after the tenth call, for the eleventh call to arrive$) =$ _______

b) Let N_t be the number of calls arriving in the first t seconds. Simplify exactly (of course without using a calculator) the ratio $\mathbb{P}(N_t = 5)/\mathbb{P}(N_t = 4) =$ _______.

d1) Simplify $\mathbb{P}(N_{50} = 7) =$ _______.

d2) Simplify $\mathbb{P}(N_{20} = 3) =$ _______.

e) Simplify, to show binomial coefficients in the answer: $\mathbb{P}(N_{20} = 3 | N_{50} = 7) =$ _______.

Short answers Work in seconds; the rate is $r = .2$. a1) $\mathbb{P}(W_1 > t) = e^{-rt} = e^{-.8}$. a) $\mathbb{P}(W_{11} > t) =$ same answer as a1). b) With $\lambda = rt = .2t$, $\lambda/5 = .04t$. d1) Here $rt = 50r = 10$, so $e^{-10}10^7/7!$. d2) Now $rt = 20r = 4$ so $e^{-4}4^3/3!$ e) Use (2), with $p = 20/50 = 2/5$, to get $\binom{7}{3}(2/5)^3(3/5)^4$. (This was a numerical example to help you understand (4), below.)

407 digression; some proofs. You do *not* need to learn these for 407.

First, how is Poisson as the limit of Binomial distributions proved? Some students know, from their calculus study, that $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$. A better lemma is: if $c_1, c_2, \ldots$ have $n\, c_n \to \ell \in (-\infty, \infty)$, then $(1 + c_n)^n \to e^{\ell}$. For the Binomial distribution, this gets applied with $c_n = -p_n$ and

$\ell = -\lambda < 0$, but the result about $(1+c_n)^n$ is very useful both for positive and for negative values of the limit ℓ. For the proof: by the continuity of log and exponential function, the result is equivalent to showing $\ln(1+c_n)^n \to \ell$. Equivalently, $n\ \ln(1+c_n) \to \ell$, which, with our notation for asymptotics, is that $c_n \sim \ell/n$ implies $\ln(1+c_n) \sim \ell/n$. Note that $nc_n \to \ell \in (-\infty, \infty)$ implies that $c_n \to 0$. The statement that $f(x) := \ln(1+x)$ has $f'(x) = 1/(1+x)$ and hence $f'(0) = 1$ says that if $c_n \to 0$ with $c_n \neq 0$, then $\ln(1+c_n) \sim c_n$. Multiplying by n, $n \ln(1+c_n) \sim n\, c_n$; by the hypothesis that $n\, c_n \to \ell$, being asymptotic to $n\, c_n$ is the same as converging to ℓ. QED.

Second, the superposition theorem. Suppose that X is Poisson(μ), Y is Poisson(ν), X and Y are independent, $Z = X + Y$, and $\lambda = \mu + \nu$, and $p = \mu/\lambda$, so that $\mu = \lambda\, p$ and $\nu = \lambda(1-p)$. In the following calculation, consider events $A = \{X = k\}$, $B = \{Y = n-k\}$, and $C = \{Z = n\}$. To verify the displayed equality below, notice that $\lambda^n\, p^k (1-p)^{n-k} = (\lambda\, p)^k (\lambda(1-p))^{n-k}$, and that the factorials match up with the binomial coefficient.

$$e^{-\mu} \frac{\mu^k}{k!} \ \ e^{-\nu} \frac{\nu^{n-k}}{(n-k)!} = e^{-\lambda} \frac{\lambda^n}{n!} \ \ \binom{n}{k} p^k (1-p)^{n-k}. \tag{4}$$

The left side above arises as $\mathbb{P}(A \cap B)$, using the given Poisson distributions for X and Y, and the independence of X and Y. Summing over $k = 0$ to n proves that Z has Poisson distribution with parameter λ; this proves the superposition theorem. To prove the extended superposition result, discussed after b) in the meteor strike story: start from the right side, and the assumption that Z is Poisson(λ), and define X to have the following conditional distribution: given $\{Z = n\}$, X is Binomial(n, p) (and $Y := Z - X$); the right side expresses $\mathbb{P}(A \cap C) = \mathbb{P}(C)\mathbb{P}(A|C)$. The equality with the left side shows that X, Y are independent Poisson. Noting that $A \cap B = A \cap C \subset C$, we get $\mathbb{P}(A|C) = \mathbb{P}(A \cap C)/\mathbb{P}(C) = \mathbb{P}(A \cap B)/\mathbb{P}(C) = \mathbb{P}(A)\mathbb{P}(B)/\mathbb{P}(C)$, using the independence of A and B for the final equality. This proves (2).