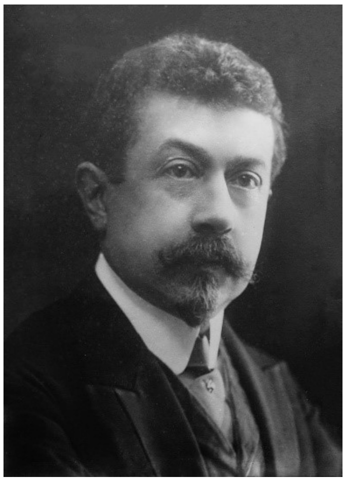
Paul Painlevé and his equations

Paul Painlevé [pɔl pɛ̃ləve]; December 5, 1863 – October 29, 1933



In 1915

In 1923

Prime Minister of France

September 12, 1917 – November 16, 1917
and then again
April 17, 1925 – November 28, 1925

The 150th anniversary of Paul Painlevé's birth was marked by the French National Assembly in December 2013.

Math Ph.D. in 1887. **Advisor:** Émile Picard. **Dissertation:** *Sur les lignes singulières des fonctions analytiques*

Mathematical background.

The main object is an ordinary differential equation for a function w of a **complex** variable z,

$$w'(z)=R(w,z) \text{ or } w''(z) = R(w,w',z),$$

with a *rational* function *R*. The solution can have fixed singular points (not depending on the initial conditions) and movable singular points (depending on the initial conditions).

The equation is said to have **Painlevé property** if the only movable singularities are poles.

Poincaré and L. Fuchs showed that any non-linear first order equation with the Painlevé property can be transformed into the Weierstrass elliptic function or the Riccati equation.

For second-order equations, there are 50 ``equivalence classes'' (equivalence modulo a linear fractional transformation $z \to (az+b)/(cz+d)$, $ad-bc\neq 0$). Of those 50, 44 are reduced to ``familiar types'' (linear, Weierstrass elliptic, etc.) and the remining six are the *Painlevé equations* P_1-P_6 (or P_I-P_{VI}).

For equations of orders greater than 2 the solutions can have moving *natural boundaries.*

The six equations

From Valerii I. Gromak · Ilpo Laine · Shun Shimomura. Painlevé *Differential Equations in the Complex Plane,* published by Walter de Gruyter, 2002.

The Painlevé equations were first derived, between 1895–1910, in the investigations by Painlevé and Gambier while studying the following problem originally posed by Picard [1]: Given $R(z, w, w')$ rational in w and w' and analytic in z, what are the second order ordinary differential equations of the form

$$w'' = R(z, w, w') \tag{0.1}$$

with the property that the singularities other than poles of any solution of (0.1) depend on the equation in question only and not on the constants of integration?

Painlevé [1] and Gambier [1] proved that there are fifty canonical equations of the form (0.1) with the property proposed by Picard. This property is known as the Painlevé property, and the differential equations, respectively difference equations, possessing this property are called equations of Painlevé type (P-type). For later presentations of this classical background, see Ince [1] and Bureau [1]. The method introduced by Painlevé to solve the classification problem by Picard was completely different from the classical Fuchs method for solving the similar problem in the case of first order differential equations. The Painlevé method relies on an application of a Poincaré theorem concerning the expansion of solutions in a series of powers of small parameters, called the α-method. By this method, finding necessary conditions for the Painlevé property is relatively easy, while finding sufficient conditions becomes more complicated.

Among the fifty equations obtained, the following six, known as the Painlevé differential equations, appear to be the most interesting ones:

$$w'' = 6w^2 + z, \tag{P_1}$$

$$w'' = 2w^3 + zw + \alpha, \tag{P_2}$$

$$w'' = \frac{(w')^2}{w} - \frac{1}{z}w' + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}, \tag{P_3}$$

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{P_4}$$

$$w'' = \frac{3w-1}{2w(w-1)}(w')^2 - \frac{1}{z}w' + \frac{1}{z^2}(w-1)^2\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \tag{P_5}$$

$$\begin{aligned} w'' = {} & \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' \\ & + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right) \end{aligned} \tag{P_6}$$

The solutions of P_I–P_{VI} are called the *Painlevé transcendents*.