**Partial Differential Equations: Basic Terms and Four Main Equation in the Whole Space**[1]

**Some notations**

- Subscripts as partial derivatives: For $u = u(t, x, y, \ldots)$,
$$u_t = \frac{\partial u}{\partial t},\ u_{xx} = \frac{\partial^2 u}{\partial x^2},\ \text{etc.}$$
- $\mathbb{R}^n$ is a collection of ordered $n$-tuples [or vectors] $(x_1, \ldots, x_n)$, with each $x_k \in \mathbb{R}$; for $n = 2$, we alternatively write $(x, y)$, and for $n = 3$, $(x, y, z)$;
- For $g = (g_1, \ldots, g_n)$ and $h = (h_1, \ldots, h_n)$,
$$g \cdot h = g_1 h_1 + \cdots + g_n h_n,\ |h|^2 = h \cdot h.$$
- $\nabla$ is the "vector"
$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right);$$
- $\boldsymbol{\Delta}$ is the Laplace operator, or Laplacian,
$$\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

**The four main equations.**

`Transport equation` $u_t + v \cdot \nabla u = 0,\ u = u(t, x),\ t > 0,\ x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ is the constant vector representing the velocity of the transport [note that $v$must have units of speed]. It is a particular case of the `equation of continuity` $u_t + \div(vu) = 0$, which, in turn, comes from conservation of "matter" and the divergence theorem.

`Heat equation` $u_t = a\boldsymbol{\Delta} u,\ u = u(t, x),\ t > 0,\ x \in G \subseteq \mathbb{R}^n$. It is a combination of the equation of continuity with Fick's Law of diffusion $vu = -A\nabla u$ in the particular case of a `linear stationary isotropic homogeneous medium`.

`Wave equation` $u_{tt} = c^2 \boldsymbol{\Delta} u,\ u = u(t, x),\ t > 0,\ x \in G \subseteq \mathbb{R}^n$. The two main sources of the wave equation are (a) Electromagnetic theory (`Maxwell's equations`) and (b) Elasticity (Newton's second law, combined with the divergence theorem and Hook's law, once again for linear stationary isotropic homogeneous medium.

`Poisson equation` $\boldsymbol{\Delta} u = -f,\ u = u(x),\ x \in G \subseteq \mathbb{R}^n$ [known as `Laplace equation` when $f = 0$]. Both equations can appear (a) In electromagnetic theory, e.g. as an alternative form of the first Maxwell's equation; (b) as the stationary regime [limiting case, as $t \to \infty$], of heat and wave equations.

**Basic Terminology**

- `Initial value problem` [IVP] or `Cauchy problem`, when $u = u(t, x)$ and we have the `initial condition(s)`, that is, the value of $u(0, x)$, and, if necessary, $u_t(0, x)$, $u_{tt}(0, x)$, etc.
- `Boundary value problem` [BVP], when $x \in G \subset \mathbb{R}^n$ and the value of $u$ on the boundary of $G$ [`boundary condition`] is prescribed.
- `Initial-boundary value problem`, when you have both initial and boundary conditions.
- `Linear equation`, when the unknown function $u$ enters in a linear way.
- `Semi-linear, quasi-linear, fully nonlinear` are various "degrees" of nonlinearity.
- `Order of the equation` is the highest order of the partial derivative appearing in the equation.
- `Elliptic, hyperbolic, parabolic` equations: once we know what an elliptic operator is [the $(\pm)$Laplace operator is an example, but the general definition is hard: basically, you want an "abstract" operator to "behave" like $\pm\boldsymbol{\Delta}$], then, if $A$ is an elliptic operator, then equations of the form $Au = \ldots$ are elliptic; $u_{tt} - Au = \ldots$, hyperbolic, and $u_t - Au = \ldots$, parabolic. In particular, heat equation is parabolic, wave equation is hyperbolic, and Laplace/Poisson is elliptic. For second-order equations in two independent variable, there is an alternative classification that somewhat resembles ellipse/hyperbola/parabola.

**Solution of the main equations in the whole space.**

**Transport equation:** if $u_t + v \cdot \nabla u = 0$ and $u(0, x) = \varphi(x)$ is a $C^1$ function [that is, continuously differentiable], then $u(t, x) = \varphi(x - vt)$. Indeed, by the chain rule, $u_t = -v \cdot \nabla\varphi,\ \nabla u = \nabla\varphi$. In words, the "initial profile" $\varphi$ is moving "to the right" with constant velocity $v$ [when $n = 1$ and $v > 0$, quotation marks can be removed].

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**Heat equation**: If $u_t = a\boldsymbol{\Delta} u$ and $u(0,x) = \varphi(x)$, then

$$u(t,x) = \frac{1}{(4\pi a t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/(4at)}\varphi(y)\,dy \tag{1}$$

Indeed, let us start with $n = 1$: $u_t = au_{xx}$. Take the Fourier transform on both sides in the $x$-variable:

$$\hat{u}(t,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t,x)e^{-\mathrm{i}x\omega}\,dx.$$

Then the properties of the Fourier transform imply $\hat{u}_t = -a\omega^2\hat{u}$ which is an ODE, with solution

$$\hat{u}(t,\omega) = e^{-a\omega^2 t}\hat{\varphi}(\omega). \tag{2}$$

As we know, for $\sigma > 0$, the Fourier transform of $e^{-x^2/(2\sigma^2)}$ is $\sigma e^{-\omega^2\sigma^2/2}$ Then, with $\sigma^2 = 2at$, we conclude that the Fourier transform of the function $K(t,x) = \frac{1}{\sqrt{2at}}e^{-x^2/(4at)}$ is exactly

$$\mathcal{F}[K](\omega) = e^{-a\omega^2 t}. \tag{3}$$

We also know that product of Fourier transforms corresponds to the convolution, with the $\sqrt{2\pi}$ factor in the right place. In other words, (2) and (3) imply

$$u(t,x) = \frac{1}{\sqrt{4\pi a t}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/(4at)}\varphi(y)\,dy, \tag{4}$$

which is (1) with $n = 1$.

The case of $n > 1$ follows after observing that the $n$-dimensional version of (3) is a product of $n$ one-dimensional versions.

The function

$$H(t,x;a) = \frac{1}{(4\pi a t)^{n/2}} e^{-x^2/(4at)}$$

is called the `heat (Gaussian) kernel`. More generally, a `kernel` refers to a function which gives you something interesting/useful/etc. after you convolve another function with it.

**Wave equation.** If $u_{tt} = c^2 u_{xx},\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$ [now the equation is second-order in time, so we need two initial conditions, corresponding to the initial displacement and initial speed] then

$$u(t,x) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy \tag{5}$$

Indeed, taking the Fourier transform, similar to the heat equation, we get $\hat{u}_{tt}(t,\omega) = -c^2\omega^2\hat{u}(t,\omega)$, or $\hat{u}(t,\omega) = \hat{\varphi}(\omega)\cos(c\omega t) + \frac{\hat{\psi}(\omega)}{c\omega}\sin(c\omega t)$. Then (5) follows from the properties of the Fourier transform, keeping in mind that

$$\cos(c\omega t) = \frac{e^{\mathrm{i}c\omega t} + e^{-\mathrm{i}c\omega t}}{2},\quad \sin(c\omega t) = \frac{e^{\mathrm{i}c\omega t} - e^{-\mathrm{i}c\omega t}}{2\mathrm{i}}$$

with multiplication by complex exponentials corresponding to translations in the physical space, and the Fourier transform of $\int_0^x \psi(y)\,dy$ is $\hat{\psi}(\omega)/(\mathrm{i}\omega)$.

Now only the case $n = 1$ is easily doable [and even then equality (5) has a special name: the `d'Alembert formula`.] The cases $n = 2$ and $n = 3$ are (surprisingly) more difficult and give very different answers.

**Poisson equation**

Here, we just take the Fourier transform on both sides to get

$$\hat{u}(\omega) = \frac{\hat{f}(\omega)}{|\omega|^2} \tag{6}$$

[now you see why we had $-f$ from the start; otherwise we would have to keep writing the $-$ sign coming from the Fourier transform of $\boldsymbol{\Delta} u$. Inverting the Fourier transform in (6) might not always be possible because of "division by zero" and, because of that, when it comes to the whole space, people prefer the following modification:

$$\boldsymbol{\Delta} v - v = -f$$

so that

$$\hat{v}(\omega) = \frac{\hat{f}(\omega)}{1 + |\omega|^2}. \tag{7}$$

Now there is no division by zero, and $v$ can always be recovered. In fact, for many purposes, the explicit formula for $v$ might not be as useful as (7).