PDEs of Musical Instruments

To varying degree of accuracy, most musical instruments can be described by the wave equation

$$u_{tt} = c^2 \boldsymbol{\Delta} u,\ t > 0,\ x \in G \subset \mathbb{R}^n$$

with the function $u = u(t, x)$ representing small deviations of a suitable quantity from the equilibrium position corresponding to $u = 0$. The boundary conditions are a major part of the story. In particular, if λ_*^2 is the smallest eigenvalue of $-\boldsymbol{\Delta}$ in G with the corresponding boundary conditions (BC), then the fundamental (lowest) frequency of the instrument is

$$\omega_* = c\lambda_*.$$

Note that this is the angular frequency; the corresponding linear frequency [measured in Hz] is

$$\nu_* = \frac{\omega_*}{2\pi}.$$

The boundary conditions are either zero Dirichlet [ZD] when $u|_{\partial G} = 0$, or mixed [MX] when both u and ∇u are involved.

The following table summarizes the results.

Instrument	c^2	G	BC	λ_*
Strings	$\frac{\text{tension}}{\text{density}}$	$[0, L]$	ZD	π/L
flute	sound in air	$[0, L]$	ZD	π/L
clarinet	sound in air	$[0, L]$	MX	$\pi/(2L)$
oboe, sax, brass	sound in air	in $\mathbb{R}^3$	MX	π/L
drums	$\frac{\text{tension}}{\text{density}}$	disk radius R	ZD	$\alpha_{1,0}/R$

Clarinet vs flute. Both are modeled by a one-dimensional object (a thin cylinder) of about the same length [about 0.7m for the flute; about 0.6m for the clarinet]. The two open ends of the flute imply zero boundary conditions for the deviation of the air density from the equilibrium, resulting in the same eigenfunctions

$$\varphi_k(x) = \sin(\pi k x/L)$$

as for the strings. In particular,

$$\lambda_* = \lambda_1 = \frac{\pi}{L}.$$

The mouth piece of the clarinet imposes the `Neumann` boundary condition $u_x(0,t) = 0$ at that end [intuitively, the density is the largest at the point of entry]. The resulting eigenvalue problem

$$\varphi_k''(x) = -\lambda_k^2 \varphi(x), \ \varphi_k'(0) = \varphi_k(L) = 0$$

leads to

$$\varphi_k(x) = \cos(\lambda_k x), \ \lambda_k L = \frac{\pi}{2} + \pi k, \ k = 0, 1, 2, \ldots,$$

so that

$$\lambda_* = \lambda_0 = \frac{\pi}{2L}.$$

Taking $c = 340m/s$ for the speed of sound, we get somewhat reasonable estimates for the lowers frequencies of the flute and the clarinet:

$$\nu_{*,f} \approx \frac{340}{2 \cdot 0.7} \approx 240Hz, \ \nu_{*,c} \approx \frac{340}{4 \cdot 0.6} \approx 140Hz$$

compared to "official" $265Hz$ and $165Hz$, respectively.

Three-dimensional but spherically symmetrical instruments.

With varying degree of accuracy, we think of the the pressure distribution in the oboe, as well as in sax, trumpet, trombone, and tuba, as spherically symmetric, when the distance r is measures from the corresponding mouthpiece. The main reason for going three-dimensional is the bell-shaped opening at the end of each of these instruments.

Remembering that, in three dimensions, the Laplacian of a spherically symmetric function $g = g(r), \ r = \sqrt{x^2 + y^2 + z^2}$, is

$$\boldsymbol{\Delta} g(r) = \nabla \cdot \nabla g(r) = \nabla \cdot \left(\frac{g'(r)}{r} \vec{r} \right) = \frac{rg''(r) - g'(r)}{r^2} \frac{\vec{r} \cdot \vec{r}}{r} + \frac{g'(r)}{r} \nabla \cdot \vec{r} = g''(r) + \frac{2g'(r)}{r},$$

we end up with the eigenvalue problem

$$\varphi_k''(r) + \frac{2\varphi_k'(r)}{r} = -\lambda_k^2 \varphi_k(r)$$

with the boundary conditions $\varphi_k'(0) = \varphi_k(L) = 0$: we now know that a mouthpiece means Neumann boundary condition, and the above equation confirms it; with $\varphi_k'(0) \neq 0$, the second term will blow up.

A substitution

$$\psi_k(r) = r\varphi_k(r)$$

leads to the equation

$$\psi_k''(r) = -\lambda_k^2 \psi_k(r)$$

[check it: $\varphi = \psi/r, \ \varphi' = \psi'/r - \psi/r^2$, etc.] with *zero Dirichlet boundary conditions* $\psi_k(0) = \psi_k(L) = 0$: because $\varphi_k(0)$ is a [finite] number, we have to have $\lim_{r \to 0+} r\varphi_k(r) = 0$. In other words, we get the same eigenvalues as for the strings and the flute!

Now, what about the bassoon and the French horn?

Mathematics of the drums.

As a mathematical object, a drum is a (bounded) sub-set G of the plane $\mathbb{R}^2$ that is not too bad; as always, the precise meaning of "bad" depends on the problem you are trying to solve. Most of the time, it is enough to assume that the boundary ∂G of G is not too "wild" and, for example, can be written as a level set of a "nice" function of two variables or otherwise is a reasonable curve so that we can use Green's formula.

The problem we are trying to solve is the wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}), \ t > 0, \ (x, y) \in G, \tag{1}$$

with zero boundary conditions $u|_{\partial G} = 0$ and some initial conditions $u|_{t=0} = f(x, y),\ u_t|_{t=0} = g(x, y)$.

Here is the general solution of the problem.

Note that the operator

$$A : u(x, y) \mapsto -\mathbf{\Delta} u(x, y) \equiv u_{xx}(x, y) + u_{yy}(x, y)$$

defined on twice continuously differentiable functions $u = u(x, y),\ (x, y) \in G$, satisfying $u|_{\partial G} = 0$, is symmetric and positive definite. Indeed, keeping in mind that

$$\nabla \cdot (v \nabla u) = \nabla u \cdot \nabla v + v \mathbf{\Delta} u$$

we compute

$$\begin{aligned} -(Au, v) &= \int_G (\mathbf{\Delta} u(x, y)) v(x, y) dx dy = \int_G \big(\nabla u \cdot \nabla v - \nabla \cdot (v \nabla u)\big) dx dy \\ &= \int_G \big(\nabla u \cdot \nabla v) dx dy - \int_{\partial G} v(\nabla u \cdot \vec{n}) d\ell; \end{aligned}$$

in the third equality we use the divergence form of Green's formula so that $\vec{n}$ is the outside unit normal to ∂G. The boundary conditions imply that the last term [the line integral] is equal to zero, that is,

$$-(Au, v) = \int_G (\nabla u \cdot \nabla v) dx dy = -(Av, u);$$

the second equality follows after exchanging u and v and repeating the computations. In other words, the operator is indeed symmetric. Putting $u = v$ shows that the operator is positive-definite:

$$-(Au, u) = \int_G (\nabla u \cdot \nabla u) dx dy > 0,\ u \not\equiv 0 :$$

the only way the integral is zero is when $\nabla u \equiv 0$ in G, that is, when u is constant, which, because of the zero boundary conditions, is only possible when $u \equiv 0$ in G.

Let us now consider the eigenvalue problem

$$\mathbf{\Delta} \varphi_k = -\lambda_k^2 \varphi_k,\ \varphi_k|_{\partial G} = 0. \tag{2}$$

We already know that, because of symmetry and positivity, $\lambda_k^2 > 0$ and if $\lambda_k \neq \lambda_n$, then

$$\int_G \varphi_k \varphi_n dx dy = 0. \tag{3}$$

It turns out that, just as for symmetric matrices, all eigenvalues are non-defective (algebraic multiplicity is the same as geometric multiplicity) and the corresponding eigenfunctions $\varphi_k,\ k \geq 1$ can be chosen so as to form an orthonormal basis in $L_2(G)$. That is, (3) holds for all $k \neq n$ and if $h = h(x, y)$ is a function on G such that

$$\int_G |h|^2 dx dy < \infty,$$

then

$$h(x, y) = \sum_k \left(\int_G h(z, w) \varphi_k(z, w) dz dw \right) \varphi_k(x, y).$$

After that, usual separation of variables leads to the solution of equation (1):

$$\begin{aligned} u(t, x, y) &= \sum_k \left(f_k \cos(c\lambda_k t) + \frac{g_k}{c\lambda_k} \sin(c\lambda_k t) \right) \varphi(x, y), \\ f_k &= \int_G f(x, y) \varphi_k(x, y) dx dy,\ g_k = \int_G g(x, y) \varphi_k(x, y) dx dy. \end{aligned} \tag{4}$$

A famous question, "Can we hear the shape of the drum," posed by M. Kac [of the Feynman-Kac formula] asks whether the collection of numbers $\lambda_k,\ k \geq 1$, uniquely determines the domain G. The general answer is "No" [with an explicit construction of two different domains with the same collection of λ_k, but, as always, it is just the beginning of an interesting story; check it out.

Our next task is to compute the numbers λ_k for the disk of radius R:

$$G = \{(x, y) : x^2 + y^2 < R^2\},$$

with disk being the shape of a typical drum.

To this end, we switch to polar coordinates

$$x = r\cos\theta,\ \ y = r\sin\theta$$

and then

$$\boldsymbol{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

We also separate the variables, looking for solutions of (2) in the form

$$\varphi_k(x, y) = F(r)H(\theta).$$

Then (2) becomes

$$r^2F''(r) + rF'(r) + (r^2\lambda_k^2 - A)F(r) = 0,\ \ H''(\theta) + AH(\theta) = 0$$

for some number A.

In the equation for F, we change the variable

$$r = \frac{z}{\lambda_k}$$

to get the following equation for $V(z) = F(z/\lambda_k)$:

$$z^2V''(z) + zV'(z) + (z^2 - A)V(z) = 0 \tag{5}$$

which we recognize as the Bessel equation. It is totally clear that the function φ_k must NOT have any singularities at zero $r = 0$: there is nothing in the original equation (2) to suggest otherwise. Then (5) must have a solution that has no singularities at $r = 0$, which, by the Fuchs-Frobenius theory, can only happen when $A = N^2$ for a non-negative integer N. The corresponding solution is, up to a constant factor, Bessel's function

$$V(z) = J_N(z) = \sum_{j=0}^{\infty}(-1)^j\frac{z^{2j+N}}{2^{2j+N}j!(j+N)!}. \tag{6}$$

For each N, we have two "essentially different" choices for the function H:

$$H_N(\theta) = \cos(N\theta),\ \ H_N(\theta) = \sin(N\theta).$$

Note that there is an alternative way to conclude that $A = N^2$, by noticing that the function H must be 2π-periodic.

We still have not figured out λ_k, but we have yet to use the boundary condition $\varphi_k|_{r=R} = 0$ which becomes $F(R) = 0$ or $V(R\lambda_k) = 0$ or

$$J_N(R\lambda_k) = 0.$$

Now we are facing yet another problem: showing that the function $J_N = J_N(z)$ has at least one zero for $z > 0$. It turns out that, for each N, the function J_N has infinitely many positive zeros $\alpha_{m,N}$, and each one of the zeros is simple and is not shared by any other J_M. In particular, $\alpha_{1,0} \approx 2.4$.

A "preponderance of evidence" argument justifying some of these claims is the alternating sing in the power series expansion of J_N, suggesting the same type of behavior as sine or cosine. On the other hand, power series of e^{-x^2} is also alternating, so there is some reasonable doubt. Accordingly, a complete proof proceeds as follows. In (5) we make a change of the unknown function, $W(z) = \sqrt{z}\,V(z)$ to conclude that

$$W''(z) + \left(1 - \frac{N^2 - (1/4)}{z^2}\right) W(z) = 0,$$

and then using the `Sturm comparison theorem` to conclude that, for large z, the function W is indeed "similar" to $\sin(z)$ or $\cos(z)$.

We can now summarize the results about (2) when G is a disk of radium R. Let $\alpha_{m,N} > 0$ denote root number m of the function J_N: $J(\alpha_{m,N}) = 0$, $m = 1, 2, \ldots$, $N = 0, 1, 2 \ldots$ [Note that $J_N(0) = 0$ for $N = 1, 2, \ldots$]. Then, for $k = (N, m)$,

$$\lambda_k = \frac{\alpha_{m,N}}{R}, \quad \varphi_k = C_k J_N(\alpha_{m,N} r/R) H_N(\theta)$$

for some constant C_k to ensure $\int_G |\varphi_k|^2 dxdy = 1$. In particular,

(1) the collection of eigenvalues and eigenfunctions is indexed by a two-dimensional array (N, m), which, if necessary, can be converted into a one-dimensional array [a standard CS exercise];
(2) for $r \in [0, R]$ and every N, the function $J_N(\alpha_{m,N} r/R)$ has exactly m zeros.
(3) A rather mysterious identity

$$\int_0^R r J_N(\alpha_{m,N} r/R) J_N(\alpha_{\ell,N} r/R) dr = 0, \ \ell \neq m,$$

is actually a particular case of (3) if you remember that $dxdy = rdr\, d\theta$.
(4) The *basic* nodal lines on the drum are the points where $\varphi_k = 0$, that is, either straight lines $\theta =$const or circles $r =$const. Those are the places you do not want to hit while playing the drum. Sometimes, you can even see some of those lines on a well-used drum [e.g. timpani in Disney Concert Hall].

Many of the above computations are rather general and apply in any number of dimensions and to both heat and wave equations. In particular, the same Bessel functions can describe the heat distribution in a sauce pan or in a pipe line.

Equation (2) is sometimes called the `Helmholtz equation`. For a nice bounded domain $G \subset \mathbb{R}^n$, the numbers λ_k obey the `Weyl Law`:

$$\lim_{k \to \infty} \frac{\lambda_k}{k^{1/n}} = C_G$$

for some number C_G depending on the domain G.