A summary of One-Dimensional Diffusions[1]

The Definition. A diffusion $X = X(t),\ t \geq 0$, on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \Omega)$ with the usual conditions, is a *strong Markov family* with continuous samples paths in the state space $\mathbf{I}\bigcup\{\triangle\}$, where $\mathbf{I}$ an interval of one of the forms $[L, R)$, $(L, R]$, $[L, R]$, or (L, R), with L and/or R possibly infinite, and $\triangle$ is an additional point not in $\mathbb{R}$ and corresponds to termination of the process. In particular, an extended half-line $[0, +\infty]$ is a possible state space.

Main notations

- $\zeta = \inf\{t > 0 : X(t) = \triangle\}$, with convention $X(t) = \triangle, t \geq \zeta$ [termination time];
- $\tau_\alpha(X) = \inf\{t > 0 : X(t) = \alpha\}$, $\alpha \in \mathbb{R}$ [hitting time];
- $\tau_{\alpha,\beta}(X) = \inf\{t > 0 : X(t) \notin (\alpha, \beta)\} \equiv \min(\tau_\alpha(X), \tau_\beta(X)),\ L < \alpha < \beta < R$;
- $\tau^\beta_\alpha(X) = \inf\{t > \tau_\alpha(X) : X(t) = \beta\},\ \alpha, \beta \in \mathbf{I}$;
- $\mathbb{P}^X_x(A) = \mathbb{P}\big(A|X(0) = x\big)$ [this family of probabilities is part of the definition of the diffusion], and $\mathbb{E}^X_x$ denotes the corresponding expectation;
- $W = W(t),\ t \geq 0$ [a standard Brownian motion].

The diffusion process X is called

- `Conservative` if $\mathbb{P}^X_x(\zeta = +\infty) = 1$ for all $x \in \mathbf{I}$ (the process is never terminated).
- `Regular` on $\mathbf{I}$ if $\mathbb{P}^X_x\big(\tau_\alpha(X) < \infty\big) > 0$ for all $x, \alpha \in \mathbf{I}$.
- `Recurrent` on $\mathbf{I}$ if $\mathbb{P}^X_x\big(\tau^\beta_\alpha(X) < \infty\big) = 1$ for all $\alpha, \beta, x \in \mathbf{I}$; furthermore, if $\mathbb{E}^X_x\big(\tau^\beta_\alpha(X)\big\langle\infty$, then the diffusion is `positive recurrent`, and `null recurrent` means recurrent but not positive recurrent.

Theorem [Dynkin]. Let $X = X(t),\ t \geq 0$, be a strong Markov process with values in $\mathbf{I}$ that is continuous in probability and has cadlag trajectories. If, for every $\varepsilon > 0$,

$$\lim_{h\searrow 0} \frac{1}{h}\mathbb{P}\big(|X(t + h) - X(t)| > \varepsilon|X(t) = x\big) = 0$$

uniformly in x on compact subsets of $\mathbf{I}$ and uniformly in t on compact subsets of $[0, +\infty)$, then X has a continuous modification.

Local Characterization. $X = X(t),\ t > 0$, is moving so that, for every $L < x < R$ and $t < \zeta$,

$$\mathbb{E}\Big(X(t+h) - X(t)|X(t) = x\Big) = b(t, x)h + o(h),\ \ \mathbb{E}\Big(\big(X(t + h) - X(t)\big)^2|X(t) = x\Big) = \sigma^2\big(t, x\big)h + o(h),\ h \to 0,$$

and

$$\mathbb{P}\big(t < \zeta < t + h|X(t) = x\big) = k(t, x)h + o(h),\ h \to 0,$$

for suitable non-random functions $b = b(t, x)$ [`drift`], $\sigma = \sigma(t, x)$ [`diffusion`], and $k = k(t, x)$; for a conservative diffusion, $k \equiv 0$. The diffusion is called `time homogenous` if the functions b, σ, and k do not depend on time.

Boundary behavior makes a difference: If $W = W(t),\ t \geq 0$ is a standard Brownian motion, $\mathbf{I} = [0, +\infty)$, and $a > 0$, then $X(t) = a + W(t)$ has $b(t, x) = 0$ and $\sigma(t, x) = 1$ regardless of what happens once $X(t)$ hits zero (and there are a lot of options: reflection, absorbtion, termination, etc.)

More precisely, the *local generator* of X is $\mathcal{L} : f(x) \mapsto f''(x)/2$, defined for functions that are twice-continuously differentiable on $[0, +\infty)$. The boundary behavior of X translates into the boundary conditions satisfied by the functions in the domain of the generator. The most general boundary condition has the form

$$\alpha f(0) - \beta f'(0) + \gamma f''(0) = 0,\ \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ \ \alpha + \beta + \gamma = 1, \tag{1}$$

and is known as the `Feller boundary condition`. In particular,

(1) $f(0) = 0$ corresponds to termination of X upon hitting zero;
(2) $f'(0) = 0$ corresponds to instantaneous reflection so that the resulting process is $|a + W(t)|$;
(3) $\alpha f(0) = \beta f'(0)$ corresponds to *elastic Brownian motion*: termination happens after a few random numbers of hits of 0, with instantaneous reflection if there is no termination, so that $\mathbb{P}(\zeta > t) = \exp(-\alpha L^{-a}(t)/\beta)$, where $t \mapsto L^x(t)$ is the local time process of W at the point x;
(4) $f''(0) = 0$ corresponds to absorbtion at 0: $X(t) = 0$ for all $t > \tau_{-a}$, where $\tau_{-a} = \inf\{t > 0 : W(t) = -a\}$;
(5) $\beta f'(0) = \gamma f''(0)$ corresponds to `sticky` or slow reflection, and in this case the process $X = X(t)$ is the unique weak solution of the equation[2] $dX(t) = \mu I(X(t) = 0) + I(X(t) > 0)dW(t),\ X(0) = a$, with $\mu = \beta/(2\gamma)$;

[1] Sergey Lototsky, USC
[2] this equation has no strong solutions

(6) $\alpha f(0) = -\gamma f''(0)$ means that, upon hitting zero for the first time, the process X is terminated not immediately but after staying at zero for some time that is independent of W and has exponential distribution.

Global Characterization.

For "reasonable" functions b and σ, a process $X = X(t),\ t \geq 0$, is a conservative diffusion if and only if one of the following conditions holds:

- X is the *dynamical system* defined by the family of weak solutions of the equations

(2)
$$dX = b\big(t, X(t)\big)dt + \sigma\big(t, X(t)\big)dW(t),\ \ X(0) = x \in (L, R).$$

  If, for all $x \in (L, R)$,

(3)
$$\sigma^2(x) > 0 \ \text{ and there is an } \varepsilon > 0 \ \text{ so that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(y)|}{\sigma^2(y)}\, dy < \infty,$$

  then equation (2) has a unique weak solution that is defined as long as $X(t) \in \mathbf{I}$.
- For every $\lambda \in \mathbb{R}$, the process
$$t \mapsto \exp\left(\lambda X(t) - \lambda \int_0^t b\big(s, X(s)\big)\, ds - \frac{\lambda^2}{2} \int_0^t \sigma^2\big(s, X(s)\big)\, ds\right)$$
  is a martingale [Stroock and Varadhan];
- For every twice continuously differentiable function $F = F(x),\ L < x < R$, the process
$$t \mapsto F\big(X(t)\big) - F\big(X(0)\big) - \int_0^t \left(b\big(s, X(s)\big)F'\big(X(s)\big) + \frac{1}{2}\sigma^2\big(s, X(s)\big)F''\big(X(s)\big)\right) ds$$
  is a martingale [Itô-Dynkin].

Note that the precise meaning of "reasonable" can be different in each case.

On the other hand, the process $t \mapsto \sqrt{|W(t)|}$ is a continuous, strong Markov process, but NOT a semi-martingale.

Further Results

Canonical form of the generator [Feller]. Let $X = X(t),\ t \geq 0$, be a weak solution of (2). Then, as a Markov process, it has the generator

(4)
$$\mathcal{L} : f(x) \mapsto b(x)f'(x) + \mathfrak{a}(x)f''(x), \ \ \mathfrak{a}(x) = \frac{\sigma^2(x)}{2},$$

with the suitable boundary conditions. The `formal adjoint` of $\mathcal{L}$ is
$$\mathcal{L}^* : f \mapsto -\big(b(x)f(x)\big)' + \big(\mathfrak{a}(x)f(x)\big)''.$$

For differentiable functions f, g define $(D_g f)(x) = f'(x)/g'(x)$. Then, by direct computation,

(5)
$$\mathcal{L}f = D_{\mathfrak{m}}\ D_p f,$$

where
$$p(x) = \int \left(\exp\left(-\int \frac{b(x)}{\mathfrak{a}(x)}\, dx\right)\right) dx$$

is the `scale function` and
$$\mathfrak{m}(x) = \int \frac{\exp\left(\int \frac{b(x)}{\mathfrak{a}(x)}\, dx\right)}{\mathfrak{a}(x)}\, dx$$

is the `speed measure`. Equality (5) is known as the `canonical` or the `Feller` form of the generator and clarifies many different formulas related to one-dimensional diffusions; any version of the antiderivatives in the definitions of p and $\mathfrak{m}$ will work. The following notations are also convenient:

(6)
$$s(x) = \exp\left(-\int \frac{b(x)}{\mathfrak{a}(x)}\, dx\right), \ m(x) = \frac{1}{\mathfrak{a}(x)s(x)}$$

so that

(7)
$$p(x) = \int s(x)\, dx, \ \mathfrak{m}(x) = \int m(x)\, dx, \ \frac{s'}{s} = -\frac{b}{\mathfrak{a}}.$$

Then we can easily verify (5):
$$D_{\mathfrak{m}}\ D_p f = \frac{1}{m}\left(\frac{f'}{s}\right)' = \frac{f''}{ms} - \frac{f's'}{ms^2} = \mathfrak{a}f'' - \frac{\mathfrak{a}f's'}{s} = \mathfrak{a}f'' + bf'.$$

Note that

- *Scale function removes the drift.* Indeed, because $D_p p = 1$, it follows that $\mathcal{L}p = 0$, so that if X satisfies $dX = b(X)dt + \sigma(X)dW$ and $Y(t) = p(X(t))$, then $dY = \tilde{\sigma}(Y)dW$, with $\tilde{\sigma}(y) = s\big(p^{-1}(y)\big)\sigma\big(p^{-1}(y)\big)$, where p^{-1} is the inverse function for p. As a result, under the assumption (3), the unique weal solution of (2) is given by
$$X(t) = p^{-1}\Big(p(x) + V\big(T(t)\big)\Big),\ T(t) = \inf\left\{s > 0 : \int_0^s \frac{dr}{\tilde{\sigma}\big(p(x)+V(r)\big)} > t\right\}$$
with some standard Brownian motion $V = V(t)$. Moreover, because $t \mapsto p(X(t))$ is a local martingale, we have $p(x) = \mathbb{E}_x^X p\Big(X\big(\tau_{\alpha,\beta}(X)\big)\Big)$ and then, from
$$\mathbb{E}_x^X p\Big(X\big(\tau_{\alpha,\beta}(X)\big)\Big) = p(\alpha)\mathbb{P}_X^x\big(\tau_{\alpha,\beta}(X) = \tau_\alpha(X)\big) + p(\beta)\mathbb{P}_X^x\big(\tau_{\alpha,\beta}(X) = \tau_\beta(X)\big),$$
we conclude that
$$\mathbb{P}_X^x\big(\tau_{\alpha,\beta}(X) = \tau_\alpha(X)\big) = \frac{p(\beta)-p(x)}{p(\beta)-p(\alpha)},\ \ \mathbb{P}_X^x\big(\tau_{\alpha,\beta}(X) = \tau_\beta(X)\big) = \frac{p(x)-p(\alpha)}{p(\beta)-p(\alpha)}. \tag{8}$$
- *Speed measure defines an invariant distribution for X* (if m can be properly normalized). Indeed, by (6), $(\mathfrak{a}m)' = -s'/s^2 = b/(\mathfrak{a}s) = bm$, and so the function $m = m(x)$ is a stationary solution of the forward Kolmogorov equation: $\mathcal{L}^* m = 0$. In particular, a recurrent diffusion process X is positive recurrent if and only if $\mathfrak{m}(R\text{-}) - \mathfrak{m}(L+) < +\infty$.
- *The operator $\mathcal{L}$ is symmetric when the measure $\mathfrak{m}$ defines the inner product*: assuming there are no boundary terms as we integrate by parts twice,
$$\int_{\mathbf{I}} (\mathcal{L}f)(x)g(x)\,m(x)dx = \int_{\mathbf{I}} f(x)(\mathcal{L}g)(x)\,m(x)dx,$$
which is easily seen from the canonical form (5).
- It is possible to extend (5) and many other related constructions to diffusion processes for which the scale function and speed measure, suitably defined, are not absolutely continuous with respect to the Lebesgue measure.

Feller's test for explosion. Let $X = X(t),\ t \geq 0$, be a conservative time homogenous diffusion with values in $\mathbf{I} = (L, R)$, $-\infty \leq L < R \leq +\infty$. We assume that, for $X(t) \in \mathbf{I}$, the process X satisfies (2) and conditions (3) hold. Define the following object:

- The scale function p and the speed measure $\mathfrak{m}$ by fixing a point $c \in \mathbf{I}$ and specifying the anti-derivatives in (6) and (7):
$$s(x) = \exp\left(-\int_c^x \frac{2b(y)}{\sigma^2(y)}\,dx\right),\ m(x) = \frac{2}{\sigma^2(x)s(x)},\ p(x) = \int_c^x s(y)\,dy,\ \mathfrak{m}(x) = \int_c^x m(y)\,dy. \tag{9}$$
- The stopping time
$$\mathfrak{s} = \inf\{t > 0 : X(t) \notin \mathbf{I}\}. \tag{10}$$
- The function
$$v(x) = \int_c^x s(y)\mathfrak{m}\,(y)\,dy,\ x \in \mathbf{I}. \tag{11}$$

Then [Feller's test for explosion]

(1) $\mathbb{P}_x^X(\mathfrak{s} = +\infty) = 1$ for all $x \in \mathbf{I}$ if and only if $v(L+) := \lim_{y\searrow L} v(y) = +\infty$ and $v(R\text{-}) := \lim_{y\nearrow R} v(y) = +\infty$.
(2) $\mathbb{P}_x^X(\mathfrak{s} = +\infty) < 1$ for all $x \in \mathbf{I}$ if and only if at least one of the limits $v(L+)$, $v(R\text{-})$ is finite.
(3) $\mathbb{P}_x^X(\mathfrak{s} < +\infty) = 1$ for all $x \in \mathbf{I}$ if and only if one of the following three conditions is fulfilled:
   - $v(L+) < +\infty$ and $v(R\text{-}) < +\infty$; in this case, $\mathbb{E}_x^X(\mathfrak{s}) < +\infty$;
   - $p(L+) = -\infty$ and $v(R\text{-}) < +\infty$;
   - $v(L+) < +\infty$ and $p(R\text{-}) = +\infty$.

The main point in the proof is to show that if the function $u = u(x)$ solves the *initial value problem* $\mathcal{L}u(x) = u(x),\ x > c,\ u(c) = 1,\ u'(c) = 0$, then, using the Itô formula [with several localizations and subsequent passages to the limit], we conclude that $t \mapsto e^{-t\wedge\mathfrak{s}}u\big(X(t\wedge\mathfrak{s})\big)$ is a non-negative super-martingale and therefore has a finite limit as $t \to \infty$. As a result, because $X(\mathfrak{s})$ is either L or R, infinite values of $u(L+)$ and $u(R\text{-})$ make finite value of $\mathfrak{s}$ impossible. Now, because the direct analysis of the function u is hard, we argue that $1 + v(x) \le u(x) \le e^{v(x)}$, by showing that $u(x) = \sum_{n\ge 0} u_n(x)$, with $u_0(x) = 1$ and $\mathcal{L}u_n = u_{n-1}, u_n(c) = u_n'(c) = 0,\ n > 0$, and noticing that, by (5), we have $v(x) = u_1(x)$.

Note that

(1) The choice of the point c does not affect the results.
(2) Existence and uniqueness of the solution of (2) imply that $\mathbb{P}_x^X(\mathfrak{s} > 0) = 1$ for all $x \in \mathbf{I}$.
(3) The actual "explosion" happens when $\mathbb{P}_x^X(\mathfrak{s} = +\infty) < 1$ and $\mathbf{I} = (-\infty, +\infty)$; otherwise, the Feller test is about reaching the points where conditions (3) might not hold and there is no guarantee to have weak existence/uniquness for (2).
(4) If $p(L+) = -\infty$, then $v(L+) = +\infty$, and if $p(R\text{-}) = +\infty$, then $v(R\text{-}) = +\infty$.

Behavior of the process before reaching the boundaries of the interval. Let $X = X(t),\ t \ge 0$, be a conservative time homogenous diffusion with values in $\mathbf{I} = (L, R),\ -\infty \le L < R \le +\infty$. We assume that, for $X(t) \in \mathbf{I}$, the process X satisfies (2) and conditions (3) hold. Let $\mathfrak{s}$ be the stopping time (10), and let $p = p(x)$ be the scale function.

(1) If $p(L+) = -\infty$ and $p(R\text{-}) = +\infty$, then, for every $x \in \mathbf{I}$,
$$1 = \mathbb{P}_x^X(\mathfrak{s} = +\infty) = \mathbb{P}_x^X(\sup_t X(t) = R) = \mathbb{P}_x^X(\inf_t X(t) = L);$$
An example is geometric Brownian motion $dX = bXdt + \sigma X dW(t)$ with $b = \sigma^2/2$, $L = 0$, $R = +\infty$: in this case, $p(x) = \ln x$, $X(t) = xe^{\sigma W(t)}$.
(2) If $p(L+) > -\infty$ and $p(R\text{-}) = +\infty$, then, for every $x \in \mathbf{I}$,
$$1 = \mathbb{P}_x^X(\sup_{t<\mathfrak{s}} X(t) < R) = \mathbb{P}_x^X(\lim_{t\nearrow\mathfrak{s}} X(t) = L);$$
An example is geometric Brownian motion $dX = bXdt + \sigma X dW(t)$ with $b < \sigma^2/2$; $L = 0$, $R = +\infty$.
(3) If $p(L+) = -\infty$ and $p(R\text{-}) < +\infty$, then, for every $x \in \mathbf{I}$,
$$1 = \mathbb{P}_x^X(\lim_{t\nearrow\mathfrak{s}} X(t) = R) = \mathbb{P}_x^X(\inf_{t<\mathfrak{s}} X(t) > L);$$
An example is geometric Brownian motion $dX = bXdt + \sigma X dW(t)$ with $b > \sigma^2/2$; $L = 0$, $R = +\infty$.
(4) If $p(L+) > -\infty$ and $p(R\text{-}) < +\infty$, then, for every $x \in \mathbf{I}$,
$$\mathbb{P}_x^X(\lim_{t\nearrow\mathfrak{s}} X(t) = L) = 1 - \mathbb{P}_x^X(\lim_{t\nearrow\mathfrak{s}} X(t) = R) = \frac{p(R\text{-}) - p(x)}{p(R\text{-}) - p(L+)};$$
An example is $X(t) = \sin\big(\arcsin(x) + W(t)\big)$: in this case, with $L = -1, R = 1$, we have $b(x) = -x/2$, $\sigma^2(x) = 1 - x^2$, $p(x) = \arcsin(x)$.

Some types of boundary points. Let $X = X(t),\ t \ge 0$, be conservative time homogenous diffusion with values in $\mathbf{I} = (L, R)$, $-\infty \le L < R \le +\infty$. We assume that $\mathbb{P}_x^X\big(\tau_y(X) < \infty\big) = 1$ for all $x, y \in \mathbf{I}$. Let $\mathfrak{b}$ denote either L or R. While the details can vary from place to place, the four basic ideas are as follows.

- `Regular` boundary point $\mathfrak{b}$ means $\mathbb{P}_x^X\big(\tau_{\mathfrak{b}}(X) < \infty\big) > 0$ and $\mathbb{P}_{\mathfrak{b}}^X(X)\big(\tau_x(X) < \infty\big) > 0$ for all $x \in \mathbf{I}$.
- `Natural` boundary point $\mathfrak{b}$ means $\mathbb{P}_x^X\big(\tau_{\mathfrak{b}}(X) < \infty\big) = 0$ for all $x \in \mathbf{I}$ and $X(t)$ with $X(0) = \mathfrak{b}$ is not defined [the process X cannot start at $\mathfrak{b}$]. In other words, a natural boundary point is not included in the state space of the process X.
- `Entrance` boundary point $\mathfrak{b}$ means $\mathbb{P}_x^X\big(\tau_{\mathfrak{b}}(X) < \infty\big) = 0$ and $\mathbb{P}_{\mathfrak{b}}^X(X)\big(\tau_x(X) < \infty\big) = 1$ for all $x \in \mathbf{I}$.
- `Exit` boundary point $\mathfrak{b}$ means $\mathbb{P}_x^X\big(\tau_{\mathfrak{b}}(X) < \infty\big) = 1$ and $\mathbb{P}_{\mathfrak{b}}^X(X)\big(\tau_x(X) < \infty\big) = 0$ for all $x \in \mathbf{I}$. In other words, $X(t) = \mathfrak{b}$ for all $t > \tau_{\mathfrak{b}}(X)$.

Ergodicity. Let $X = X(t),\ t \ge 0$, be conservative time homogenous diffusion with values in $\mathbf{I} = (L, R)$. If X is recurrent and the speed measure $\mathfrak{m}$ satisfies $\mathfrak{m}(\mathbf{I}) := \mathfrak{m}(R\text{-}) - \mathfrak{m}(L+) < +\infty$, then X positive recurrent and ergodic, with stationary/invariant distribution $\boldsymbol{\pi}(dx) = m(x)dx/\mathfrak{m}(\mathbf{I})$. In particular,

- $\lim_{t\to+\infty} \sup_{A\in\mathcal{B}(\mathbb{R})} |P_t(x, A) - \pi(A)\| = 0$ for every $x \in \mathbf{I}$, where $P_t(x, A) = \mathbb{P}_x^X\big(X(t) \in A\big)$;
- $\lim_{t\to+\infty} t^{-1} \int_0^t f\big(X(s)\big)\, ds = \int_{\mathbf{I}} f(y)\pi(dy)$ for every bounded measurable f, both with probability one and in L_1.

Study of ergodic behavior of X relies on the *local time* $L^x(t)$, which is now defined according to the constructions for Markov processes, as opposed to semi-martingales [partly because a diffusion might not be a semimartingale: remember $\sqrt{|W(t)|}$]. In particular, time is measured in the usual way, but the integration in space is with respect to the speed measure:

$$L^x(t) = \lim_{\varepsilon\downarrow 0} \frac{\int_0^t I\big(x-\varepsilon < X(s) < x+\varepsilon\big)\,ds}{\mathfrak{m}(x+\varepsilon)-\mathfrak{m}(x-\varepsilon)}, \quad \int_0^t f\big(X(s)\big)\,ds = \int_{\mathbf{I}} L^x(t) f(x) m(x)\,dx.$$

The Special Case of $\mathbf{I} = \mathbb{R}$.

Assume that (3) holds for all $x \in \mathbb{R}$ and let $X = X(t)$ be the unique weak solution of (2).

Khasminskii's criterion for non-explosion. Assume there exists a function $F = F(x)$ such that, for all $x \in \mathbb{R}$,

- $F(x) \geq 0$, $\lim_{|x|\to+\infty} F(x) = +\infty$,
- F is twice continuously differentiable for all x;
- There exists a real number γ such that, for all $x \in \mathbb{R}$, $(\mathcal{L}F)(x) \leq cF(x)$.

Then $\mathbb{E}_x^X F\big(X(t)\big) \leq e^{\gamma t} F(x)$. In particular, for every $T > 0$ and $x \in \mathbb{R}$, $\mathbb{P}_x^X\big(\sup_{0<t<T} |X(t)| < \infty\big) = 1$. The proof is an application of the Itô-Dynkin formula to $e^{-\gamma(t\wedge\tau_{-n,n}(X))} F\Big(X\big(t\wedge\tau_{-n,n}(X)\big)\Big)$. Note that

(1) The result only involves the generator of X and therefore extends to any number of dimensions $\mathbb{R}^{\mathrm{d}}$, and also to open sub-sets of $\mathbb{R}^{\mathrm{d}}$.
(2) The "bigger" the function F, the better the resulting integrability of X. For example, if $b(x) = x - x^3$ and $\sigma(x) = x$, then $F(x) = x^2$ works [with $\gamma = 3$], but $F(x) = e^{x^2/2}$ also works.
(3) If $2xb(x) + \sigma^2(x) \leq \gamma(1+x^2)$, then $F(x) = 1 + x^2$ works.
(4) The dual version is that if $F = F(x)$ is a *bounded* non-negative twice continuously differentiable function and $(\mathcal{L}F)(x) \geq \gamma F(x)$, then the explosion time $\mathfrak{s}$ is finite with positive probability.

Ergodicity. Define the scale function p and the speed measure $\mathfrak{m}$ according to (9) [often, $c = 0$ is the default choice]. Then

(1) The process X is recurrent if and only if

$$\lim_{x\to-\infty} p(x) = -\infty \quad \text{and} \quad \lim_{x\to+\infty} p(x) = +\infty. \tag{12}$$

(2) A recurrent process X is positive recurrent if and only

$$\mathfrak{m}(\mathbb{R}) < +\infty, \tag{13}$$

and is null recurrent if and only if $\mathfrak{m}(\mathbb{R}) = +\infty$.
(3) If the function $\sigma^2 = \sigma^2(x)$ is constant (does not depend on x), then (13) implies (12). In general, (13) does not imply (12); the standard example is $b(x) = x(1+x^2)$, $\sigma^2(x) = (1+x^2)^2$, when $p(\pm\infty) = \pm\pi/2$ and $\mathfrak{m}(\mathbb{R}) = \pi$.

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