Math 245

Practical Examples of First-Order ODEs.

POPULATION MODELS.

The basic model: $n'(t) = rn(t)$, n is current population (not necessarily humans!), r is the *growth rate*; the particular form of r determines the particular model:

(1) Elementary: $r = B - M$, B is birth rate, M is mortality rate, both are constant; first suggested by a British economist Thomas Malthus (1766–1834) around 1800.

(2) Logistic: $r = B\left(1 - \dfrac{n}{\lambda}\right)$ (birth rate is constant but mortality rate is proportional to the current population); first suggested by Belgian mathematician P. F. Verhulst (1804–1849) around 1840 and allowed him to predict the US population 100 years later to within 1% error.

(3) Logistic with threshold: $r = -r_0\left(1 - \dfrac{n}{\lambda}\right)\left(1 - \dfrac{n}{\mu}\right)$.

(4) Gompertz: $r = r_0(\ln\lambda - \ln n)$, first suggested by a British mathematician/actuary Benjamin Gompertz (1779–1865) around 1825.

(5) Doomsday: $r = r_0 n^{\gamma}$, $0 < \gamma < 1$. Proposed by H. von Foerster, P. M. Mora, and L. W. Amiot in the paper *Doomsday: Friday, 13 November, A.D. 2026*, Science 132 (November 1960): 1291–1295.

(6) Nurgaliev: $r = bn - M$ (now, mortality is constant but the birth rate is proportional to the current population).

CAPSTAN EQUATION (Euler 1769, Eytelwein 1808; Johann Albert Eytelwein (1764–1849) was a German civil engineer.)

A rope is winding around a capstan (a pole). Then

$$T_L = T_H e^{\mu\varphi},$$

where T_L is the load, T_H is the force needed to balance the load at the other end, μ is the (dimensionless) coefficient of static friction between the rope and the surface [typically between 0.2 and 0.8], and φ is the winding angle (in radians). **With five complete turns ($\varphi = 10\pi$) and $\mu = 0.6$, two pounds will balance an aircraft carrier.**

The reason: infinitesimal balance of forces, T (tension), N (normal reaction) and μN (friction) at the point of rope-pole contact. In the normal direction to the surface of the pole, $\triangle N \approx T\triangle\varphi$; in the tangential direction (along the rope), $\mu\triangle N \approx \triangle T$. After eliminating N, get $\triangle T \approx \mu T\,\triangle\varphi$ or, as a differential equation for $T = T(\varphi)$,

$$T'(\varphi) = \mu T(\varphi),\ T(0) = T_H.$$

NEWTON'S LAW OF COOLING. If $T(t)$, $t \geq 0$, is the temperature of the object at time t and $T_o < T(0)$ is the temperature of the environment, then

$$T'(t) = -\kappa\big(T(t) - T_o\big)$$

for some $\kappa > 0$.

GENERAL PROBLEM ON THE RATE OF CHANGE: "rate of change" = "rate in" − "rate out". Two standard examples: mixing and loan payment.

MIXING. Assume that IN a container comes something (call it "pollutant") as a perfect mixture at a (constant) rate a [volume/time] and with (constant) concentration ρ [mass/volume], and OUT of the container comes a perfect mixture at a (constant) rate b [volume/time]. Then the mass $m = m(t)$ of the "pollutant" at time t satisfies

$$m'(t) = \rho a - \frac{m(t)}{V_0 + (a-b)t}\, b,$$

where V_0 is the initial volume of the mixture in the container: the concentration of the pollutant in the container at time t is $m(t)/V(t)$, where $V(t) = V_0 + (a-b)t$ is the current volume of the mixture in the container. If $a \neq b$, then the process will continue until either $V_0 = (b-a)t$ (if $b > a$ and the container becomes empty) or $V_0 + (a-b)t = V_{max}$ (if $a > b$ and the container is filled to capacity V_{max}). If $a = b$, then

$$m'(t) = \rho a - \frac{m(t)}{V_0}\, a$$

and the process can continues for ever.

LOAN PAYMENT. If m is the amount you owe, r is interest rate (continuously compounded), T is the duration of the loan, and p is the payment rate (dollars per unit time), then, with "rate in"$= rm(t)$, "rate out"$= p$,

$$m'(t) = rm(t) - p;$$

here the *terminal* condition $m(T) = 0$, together with the initial amount borrowed $m(0)$, determines the payment p.

CATENARY: the shape of a hanging cable. If $y = y(x)$ is the shape, T_0 is the horizontal tension at the lowest point $(0, y_0)$, T is the tension at a point $\big(x, y(x)\big)$, m is the mass of the piece from $(0, y_0)$ to $\big(x, y(x)\big)$, ρ is the mass of the cable **per unit length**, g is the gravitational constant, then

$$T\cos\theta = T_0 \ \text{(horizontal balance of forces)}; \ T\sin\theta = mg \ \text{(vertical balance of forces)}$$

$$m = \rho \int_0^x \sqrt{1 + \big(y'(r)\big)^2}\, dr \ \text{(mass is density times arc length)},$$

and, keeping in mind that $\tan\theta = y'(x)$,

$$y'(x) = \frac{\rho g}{T_0} \int_0^x \sqrt{1 + \big(y'(r)\big)^2}\, dr.$$

Setting $a = \rho g/T_0$ and $u = y'$, we get $u' = a\sqrt{1+u^2}$, so that $u(x) = \sinh(ax)$ and

$$y(x) = y_0 + \frac{1}{a}\Big(\cosh(ax) - 1\Big);$$

note that T_0 has the units of force and therefore $1/a = T_0/(\rho g)$ has the units of length, as it should.

A separable ODE

$$f(y)y'(x) = g(x) \ \Rightarrow \ f(y)dy = g(x)dx \ \Rightarrow \ \int_{y_0}^{y} f(u)du = \int_{x_0}^{x} g(v)dv;$$

integrate, and solve for y (if possible).

All of the above application examples lead to separable equations. In particular, for some of the population models with $t_0 = 0$, $n(0) = n_0$,

$$n' = (B - M)n, \ \ n(t) = n_0 e^{(B-M)t} \ \text{(simple)};$$

$$n' = B\left(1 - \frac{n}{\lambda}\right) n, \ n(t) = \frac{\lambda n_0}{n_0 + (\lambda - n_0)e^{-Bt}} \ \text{(logistic)};$$

$$n' = r_0(\ln\lambda - \ln n)n, \ n(t) = \lambda \exp\left(\ln\frac{n_0}{\lambda}\, e^{-r_0 t}\right) \ \text{(Gompertz)};$$

$$n' = r_0 n^{1+\gamma}, \ n(t) = \frac{\kappa^\kappa}{(\kappa n_0^\gamma - r_0 t)^\kappa}, \ \kappa = \frac{1}{\gamma} \ \text{(Doomsday)}, \ t_D = \frac{\kappa n_0^\gamma}{r_0}.$$