Example on undetermined coefficients

$$y'' + 2y' + 2y = xe^{-x}\cos x \quad := \operatorname{Re}\left(xe^{\overbrace{(-1+i)}^{r}x}\right)$$

$\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i$

Guess $y = x(\alpha x + \beta)e^{rx}$ — α, β complex

$\alpha x + \beta$ - because of x

x - because $\lambda_1 = r$

Then take (real) part

plug in:

$$y = (\alpha x^2 + \beta x)e^{rx}$$

$$y' = (2\alpha x + \beta + r\alpha x^2 + r\beta x)e^{rx}$$

$$y'' = (2\alpha + 2r(2\alpha x + \beta) + r^2(\alpha x^2 + \beta x))e^{rx}$$

y'' — $2\alpha + 2r(2\alpha x + \beta) + r^2(\alpha x^2 + \beta x)$,

$2y'$ + $4\alpha x + 2\beta$ $2r\alpha x^2 + 2r\beta x$, (cancel e^{rx})

y + $2\alpha x^2 + 2\beta x$

$= x$ ← right-hand side xe^{rx} after cancelling e^{rx}

$$(\alpha x^2 + \beta x)\underbrace{(r^2 + 2r + 2)}_{0} = 0$$

x^2: $4r\alpha + 4\alpha = 1 \Rightarrow \alpha = \frac{1}{4(r+1)} = \frac{1}{4i} = -\frac{i}{4}$ $(r = -1+i)$

1: $2\alpha + 2\beta + 2r\beta = 0 \Rightarrow \beta = -\frac{\alpha}{r+1} = -\frac{\alpha}{i} = \frac{1}{4}$ $(r+1 = i)$ $\left(\frac{1}{i} = i\right)$

$$\Rightarrow y = \frac{x}{4}(1 - ix)e^{-x}(\cos x + i\sin x)$$

$$\Rightarrow \operatorname{Re} y = \frac{x}{4}e^{-x}(\cos x + x\sin x)$$

$\Rightarrow$ particular solution of $y'' + 2y' + 2y = xe^{-x}\cos x$

is $\boxed{y = \frac{xe^{-x}}{4}(\cos x + x\sin x)}$

[Now try to verify this!]

Same example, but without complex numbers.

$$y'' + 2y' + 2y = x e^{-x} \cos x$$

Then $y = x(ax+b)e^{-x}(A\cos x + B \operatorname{sen} x)$

$$= e^{-x}(ax^2+bx)(A\cos x + B \operatorname{sen} x)$$

$$y' = e^{-x}\Big[-(ax^2+bx)(A\cos x + B\sin x) + (2ax+b)(A\cos x + B\sin x) + (ax^2+bx)(-A\sin x + B\cos x)\Big]$$

use $(fg)'' = f''g + 2f'g' + fg''$

$$y'' = e^{-x}\Big[(ax^2+bx)(A\cos x + B\sin x) - 2\big[(2ax+b)(A\cos x + B\operatorname{sen} x) + (ax^2+bx)(-A\sin x + B\cos x)\big] + \big[2a(A\cos x + B\sin x) + 2(2ax+b)(-A\sin x + B\cos x) + (ax^2+bx)(-A\cos x - B\sin x)\big]\Big]$$

I GIVE UP!