11

A NOWHERE DIFFERENTIABLE FUNCTION

In Theorem 9.4 we gave a safe but unexciting condition which allowed us to differentiate a Fourier series term by term. What happens if we go ahead and differentiate formally term by term without imposing conditions? Then, *formally*,

$$\text{if } `f(x) \sim \Sigma a_r \exp irx' \quad \text{then} \quad `f'(x) \sim \Sigma ira_r \exp irx'.$$

But it is clear that we can have $\Sigma|a_r|$ convergent so that, by Theorem 9.2, the convergence of $\Sigma a_r \exp irx$ is trouble-free and yet $\limsup_{r\to\infty} |ra_r| = \infty$ so that the sum $\Sigma\, ira_r \exp irx$ cannot possibly converge. (For example, we could take $a_r = r^{-\frac{1}{2}}$ whenever $r = 2^{2^n}[n \geqslant 1]$, $a_r = 0$ otherwise.)

And that, apparently, is the end of the story. However, Weierstrass was able to see rather more in the remarks above and use them as a hint for his famous construction of a continuous function which was nowhere differentiable.

To understand the stir that this example caused, the reader must remember that though it was understood that a continuous function could fail to be differentiable at a point (look at $f(x) = |x|$), it was also generally believed that a continuous function must be differentiable at some point. Indeed, many advanced calculus texts carried 'proofs' of this fact (and even Galois seems to have thought himself in possession of such a proof).

(In passing let me add that the proofs were not necessarily worthless. For example some authors, essentially, proved the true, and interesting, theorem that a continuous function with only a finite number of maxima and minima must be differentiable at a dense set of points and then stated mistakenly that a continuous function can only have a finite number of maxima and minima.)

In this chapter we shall construct a version of Weierstrass's continuous nowhere differentiable function.

Example 11.1. $\sum_{r=0}^{n} (r!)^{-1} \sin((r!^2 t)$ *converges uniformly on* $\mathbb{T}$ *as* $n \to \infty$ *to* $h(t)$ *where* $h: \mathbb{T} \to \mathbb{R}$ *is a continuous nowhere differentiable function.*

In order to shorten the main proof we make some preliminary computations.

Lemma 11.2. (i) $\sum_{r=n}^{\infty}(r!)^{-1} \leqslant 2(n!)^{-1}$ *for* $n \geqslant 1$,
(ii) $|\sin x - \sin y| \leqslant |x-y|$ *for all* $x, y \in \mathbb{T}$,
(iii) *If* $K \geqslant 3$ *is an integer and* $x \in \mathbb{T}$ *then we can find* $y \in \mathbb{T}$ *such that* $K\pi^{-1} < |x-y| \leqslant 3K\pi^{-1}$ *and yet* $|\sin Kx - \sin Ky| \geqslant 1$.

Proof. (i) $\sum_{r=n}^{\infty}(r!)^{-1} \leqslant \sum_{r=n}^{\infty}(n!)^{-1}(n+1)^{-(r-n)} \leqslant \sum_{r=n}^{\infty}(n!)^{-1}2^{-(r-n)} = 2(n!)^{-1}$.
(ii) Use the mean value theorem. (iii) Observe that $\sin Kt$ takes all values between 1 and -1 in the range $(x+K^{-1}\pi, x+3K^{-1}\pi]$. If we take $y_1, y_2 \in (x+K^{-1}\pi, x+3K^{-1}\pi]$ with $\sin Ky_1 = 1$ and $\sin Ky_2 = -1$ then at least one of y_1 and y_2 must satisfy the conditions of (iii). ■

Proof of Example 11.1. The fact that $\sum_{r=0}^{n}(r!)^{-1}\sin((r!)^2 t)$ converges uniformly on $\mathbb{T}$ to a continuous function $h(t)$ follows from Theorem 9.2 (or directly from the Weierstrass M test). All that remains therefore is to prove that h is nowhere differentiable and to show this it suffices to show that h is not differentiable at any fixed point x.

Let us write

$$h_n(t) = \sum_{r=0}^{n-1}(r!)^{-1}\sin((r!)^2 t),$$

$$k_n(t) = (n!)^{-1}\sin((n!)^2 t),$$

$$l_n(t) = \sum_{r=n+1}^{\infty}(r!)^{-1}\sin((r!)^2 t),$$

so that $h_n + k_n + l_n = h$. This decomposition is the key to the construction. Now consider any integer $n \geqslant 3$. By Lemma 11.2(iii) we can find an $x_n \in \mathbb{T}$ such that

$$(\mathrm{A})_n \quad (n!)^{-2}\pi < |x - x_n| \leqslant 3(n!)^{-2}\pi,$$

yet $$(\mathrm{B})_n \quad |k_n(x) - k_n(x_n)| = (n!)^{-1}|\sin((n!)^2 x) - \sin((n!)^2 x_n)| \geqslant (n!)^{-1}.$$

Using the inequality $|x - x_n| \leqslant 3(n!)^{-2}\pi$ together with Lemma 11.2 (ii) we have

$$(\mathrm{C})_n \quad |h_n(x) - h_n(x_n)|$$

$$\leqslant \sum_{r=0}^{n-1}(r!)^{-1}|\sin((r!)^2 x) - \sin((r!)^2 x_n)| \leqslant \sum_{r=0}^{n-1}(r!)^{-1}|(r!)^2 x - (r!)^2 x_n|$$

$$= \sum_{r=0}^{n-1} r!|x - x_n| = \left((n-1)! + \sum_{r=0}^{n-2} r!\right)|x - x_n| \leqslant \left((n-1)! + \sum_{r=0}^{n-2}(n-2)!\right)|x - x_n|$$

$$= 2(n-1)!|x - x_n| \leqslant 6\pi n^{-1}(n!)^{-1}.$$

On the other hand Lemma 11.2(i) shows that

$$(\mathrm{D})_n \quad |l_n(t)| \leqslant \sum_{r=n+1}^{\infty}(r!)^{-1} \leqslant 2((n+1)!)^{-1}$$

for all $t \in \mathbb{T}$ and so

$$(\mathrm{E})_n \quad |l_n(x) - l_n(x_n)| \leqslant |l_n(x)| + |l_n(x_n)| \leqslant 4((n+1)!)^{-1}.$$

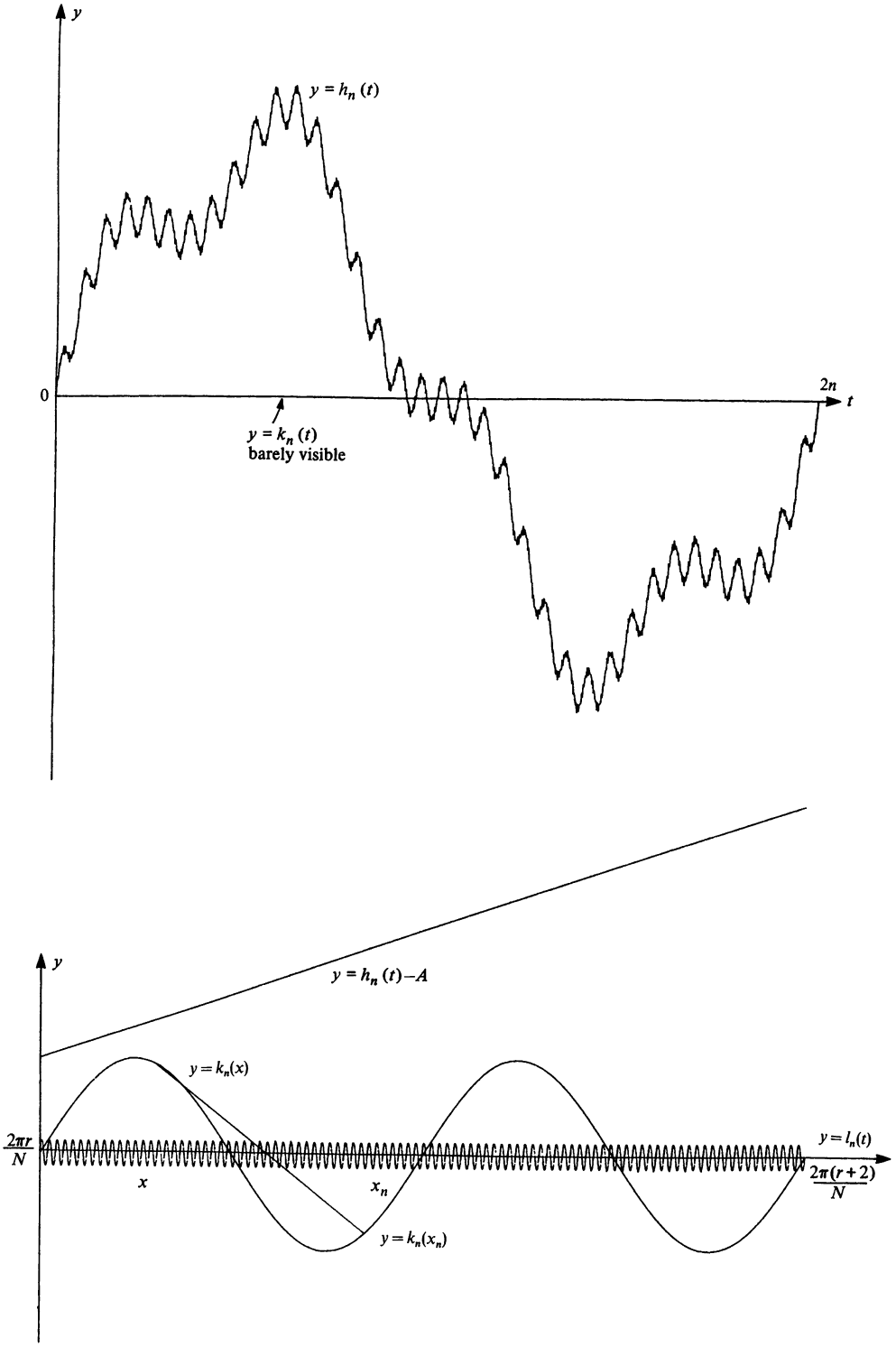

Fig. 11.1. Steps in the construction of a nowhere differentiable function.

Let us sum up what we have done so far. We have chosen an x_n whose distance from x has the same order of magnitude as the period of k_n (this is formula $(A)_n$) but for which $k_n(x) - k(x_n)$ is relatively large and so the chord joining $(x, k(x))$ to $(x_n, k_n(x_n))$ has a steep slope (this is formula $(B)_n$). Since the period of k_n is so small compared with that of h_n we know that, although h_n may be large relative to k_n, the difference $h_n(x) - h_n(x_n)$ is relatively small (this is formula $(C)_n$). Finally, since l_n is small compared to h_n, it follows that $l_n(x) - l_n(x_n)$ is small. (A look at Figure 11.1 may help a bit, reflection and the passage of time will help a great deal.)

Using the inequalities $(B)_n$, $(C)_n$ and $(E)_n$ we obtain

$$\begin{aligned}(F)_n\ |h(x) - h(x_n)| &\geqslant |k_n(x) - k(x_n)| - |h_n(x) - h_n(x_n)| - |l_n(x) - l_n(x_n)| \\ &\geqslant (n!)^{-1} - 6n^{-1}\pi(n!)^{-1} - 4((n+1)!)^{-1} \\ &= (n!)^{-1}(1 - 6\pi n^{-1} - 4(n+1)^{-1}) \geqslant (n!)^{-1}(1 - 30n^{-1}) \geqslant (n!)^{-1}/2\end{aligned}$$

whenever $n \geqslant 60$. In other words the value of

$$h(x) - h(x_n) = (k_n(x) - k_n(x_n)) + (h_n(x) - h_n(x_n)) + (l_n(x) - l_n(x_n))$$

is dominated by the $k_n(x) - k_n(x_n)$ term. Thus, since the chord joining $(x, k_n(x))$ to $(x_n, k_n(x_n))$ was steep, the chord joining $(x, h(x))$ to $(x, h(x_n))$ remains so. Using $(A)_n$ and $(F)_n$ we have, in fact,

$$(H)_n \qquad \left|\frac{h(x) - h(x_n)}{x - x_n}\right| \geqslant \frac{(n!)^{-1}}{2} \cdot \frac{1}{3\pi(n!)^{-2}} = \frac{n!}{6\pi}$$

for $n \geqslant 60$. Thus

$$\left|\frac{h(x) - h(x_n)}{x - x_n}\right| \to \infty \text{ whilst } x_n \to x,$$

so h cannot be differentiable at x. ■

Although the construction and proof above may seem hard to grasp at first, the reader who studies them carefully will find that the underlying idea is very simple. She can check her understanding by constructing a proof for Weierstrass's original function $\sum_{n=-\infty}^{\infty} a^{-n} \sin b^n x$ where b is an integer and b/a and a are sufficiently large. (The same function was discovered independently but not published by Cellérier. Many years earlier Bolzano seems to have been close to a continuous nowhere differentiable function constructed along different lines but he too did not publish.)