Bounds on the normal tail

If $X \sim \mathcal{N}(0,1)$ - standard normal

and $x > 0$ is large, then

$$\mathbb{P}(X > x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-t^2/2}\, dt$$

is small.

How small is it?

The basic bounds for $x > 0$:

$$\boxed{\frac{1}{\sqrt{2\pi}} \frac{x}{1+x^2} e^{-x^2/2} \leq \mathbb{P}(\mathcal{N}(0,1) > x) \leq \min\left(\frac{1}{2}, \frac{1}{\sqrt{2\pi}\, x}\right) e^{-\frac{x^2}{2}}}$$

Indeed, for the upper bound,

$$\int_x^{+\infty} e^{-t^2/2}\, dt = \int_x^{+\infty} t e^{-\frac{t^2}{2}} \cdot \frac{1}{t}\, dt = -\int_x^{+\infty} \frac{1}{t}\, d(e^{-\frac{t^2}{2}})$$

$$= \frac{1}{t} e^{-t^2/2} \Big|_{t=+\infty}^{t=x} - \int_x^{+\infty} e^{-t^2/2} \frac{dt}{t^2}$$

$$= \frac{e^{-x^2/2}}{x} - \int_x^{+\infty} e^{-t^2/2} \frac{dt}{t^2} < \frac{e^{-\frac{x^2}{2}}}{x}.$$

Also, $\mathbb{P}(\mathcal{N}(0,1) > x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{t^2}{2}} dt$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_x^{+\infty} e^{-\frac{t^2 - x^2}{2}}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_x^{+\infty} e^{-\frac{(t-x)(t+x)}{2}}\, dt$$

$$\underset{y = t - x}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_0^{\infty} e^{-\frac{y(y+2x)}{2}}\, dy$$

$$\underset{y+2x > y}{<} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_0^{\infty} e^{-\frac{y^2}{2}}\, dy = \frac{1}{2} e^{-\frac{x^2}{2}}.$$

For the lower bound, note that

$$\frac{d}{dx}\left(\frac{e^{-x^2/2}}{x}\right) = e^{-\frac{x^2}{2}}\left(-\frac{1}{x^2} - 1\right) = -e^{-\frac{x^2}{2}} \frac{x^2+1}{x^2}$$

Then $\int_x^{\infty} e^{-\frac{t^2}{2}} dt = \int_x^{+\infty} \frac{t^2+1}{t^2} e^{-\frac{t^2}{2}} \cdot \frac{t^2}{t^2+1}\, dt$

$$\geqslant \frac{x^2}{x^2+1} \int_x^{+\infty} \frac{t^2+1}{t^2} e^{-\frac{t^2}{2}}\, dt = \frac{x^2}{x^2+1} \cdot \frac{e^{-\frac{x^2}{2}}}{x}$$

$t \to \frac{t^2}{t^2+1}$ is increasing, $t > 0$.

$$= \frac{x}{x^2+1} e^{-x^2/2}.$$

Next step: asymtotic expansion.

from $\int_x^{+\infty} e^{-t^2/2} dt = \frac{e^{-x^2/2}}{x} - \int_x^{+\infty} e^{-t^2/2} \frac{dt}{t^2}$,

continue integrating by parts:

$$\int_x^{+\infty} e^{-t^2/2} \frac{dt}{t^2} = -\int_x^{+\infty} \frac{1}{t^3}\, d(e^{-t^2/2})$$

$$= \frac{e^{-x^2/2}}{x^3} - 3\int_x^{+\infty} e^{-t^2/2} \frac{dt}{t^4} = \ldots$$

to get

$$\int_x^{+\infty} e^{-t^2/2} dt \approx e^{-x^2/2}\left(\frac{1}{x} - \frac{1}{x^3} + \frac{1\cdot 3}{x^5} - \ldots + \frac{(-1)^n 1\cdot 3 \cdot \ldots \cdot (2n-1)}{x^{2n+1}}\right)$$

or

$$\int_x^{+\infty} e^{-t^2/2} dt \sim e^{-\frac{x^2}{2}}\left(\frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{x^{2k+1}}\right)$$

$(2k-1)!! = 1\cdot 3 \cdot \ldots \cdot (2k-1)$ (odd factorial)

not equality, because the infinite series diverges for every $x > 0$

But this is still useful!