67
UNIQUENESS FOR THE HEAT EQUATION I

When discussing the physical meaning of our solutions for the heat equation we have, more or less tacitly, assumed that they were unique. To what extent is this assumption justified?

Problem. *Suppose* $\phi:\mathbb{R}\times\mathbb{R}^+\to\mathbb{C}$ *is infinitely differentiable with*

(i) $(\partial\phi/\partial t)(x,t)=K(\partial^2\phi/\partial x^2)(x,t)$ *for all* $x\in\mathbb{R}$, $t>0$ $[K>0]$,
(ii) $\phi(x,t)\to 0$ *as* $t\to 0+$.

Does it follow that $\phi(x,t)=0$ *for all* $x\in\mathbb{R}$, $t>0$?

Our search for an answer to this problem takes us in an unexpected direction. (There is a fair amount of calculation involved but, so long as she follows the drift of the argument, the reader need not worry too much about the details.)

Lemma 67.1. (i) $x^{-n}\exp(-1/2x^2)\leqslant n^{n/2}$ *for all* $x>0$.
(ii) *Define* $h:\mathbb{R}\to\mathbb{R}$ *by*

$$\begin{aligned} h(x)&=\exp(-1/2x^2) &&\text{for } x>0,\\ h(x)&=0 &&\text{for } x\leqslant 0.\end{aligned}$$

Then h *is infinitely differentiable with*

$$\begin{aligned} h^{(r)}(x)&=Q_r(x^{-1})\exp(-1/2x^2) &&\text{for } x>0,\\ h^{(r)}(x)&=0 &&\text{for } x\leqslant 0,\end{aligned}$$

where
$$Q_r(t)=\sum_{s=0}^{r}a_{rs}t^{r+2s}$$

and
$$|a_{rs}|\leqslant 4^r r^{r-s},\ \text{for } r\geqslant s\geqslant 0, r\geqslant 1,\ \text{whilst } a_{00}=1.$$

(iii) *With* h *as in* (ii) *we have*

$$|h^{(r)}(x)|\leqslant 2^{5r}r^{3r/2}\quad\text{for all } x\in\mathbb{R}, r\geqslant 1.$$

Remark. This lemma is just a more careful reworking of Example 4.2. Although

we give a full proof below the reader may prefer to do it for herself.
Proof. (i) Setting

$$f(x) = x^{-n}\exp(-1/2x^2)$$

we see that

$$f'(x) = x^{-n-1}(-n + x^{-2})\exp(-1/2x^2),$$

so that f' is increasing for x running from 0 to $n^{-\frac{1}{2}}$ and decreasing for x running from $n^{-\frac{1}{2}}$ to ∞. Thus

$$f(x) \leqslant f(n^{-1/2}) \quad \text{for all } x > 0$$

(ii) We proceed inductively. Let $P(r)$ be the proposition that h is r times differentiable with

$$h^{(r)}(x) = Q_r(x^{-1})\exp(-1/2x^2) \quad \textit{for } x > 0,$$
$$h^{(r)}(x) = 0 \quad \textit{for } x \leqslant 0,$$

where
$$Q_r(t) = \sum_{s=0}^{r} a_{r,s}t^{r+2s}$$

and
$$|a_{r,s}| \leqslant 4^r r^{r-s}.$$

Suppose $P(r)$ is true for some $r \geqslant 1$. If $x < 0$ it is obvious that $h^{(r)}$ is differentiable at x with $h^{(r+1)}(x) = 0$. If $x > 0$ we see that $h^{(r)}$ is differentiable at x with

$$\begin{aligned} h^{(r+1)}(x) &= -x^{-2}Q_r'(x^{-1})\exp(-1/2x^2) + x^{-3}Q_r(x^{-1})\exp(-1/2x^2) \\ &= Q_{r+1}(x^{-1})\exp(-1/2x^2), \end{aligned}$$

where

$$Q_{r+1}(t) = -\sum_{s=0}^{r}(r+2s)a_{r,s}t^{r+1+2s} + \sum_{s=0}^{r} a_{r,s}t^{r+1+2(s+1)} = \sum_{u=0}^{r+1} a_{r+1,u}t^{r+1+2u},$$

with
$$a_{r+1,u} = -(r+2u)a_{r,u} + a_{r,u-1}$$

(taking $a_{r,-1} = a_{r,r+1} = 0$). It follows that

$$|a_{r+1,u}| \leqslant 3(r+1)|a_{r,u}| + |a_{r,u-1}| \leqslant 3(r+1)4^r r^{r-u} + r^{r+1-u} \leqslant 4^{r+1}r^{r+1-u},$$

as required.

Finally if $x = 0$ we see that

$$\frac{h^{(r)}(\eta) - h^{(r)}(0)}{\eta} = 0 \to 0 \quad \text{as} \quad \eta \to 0-,$$

whilst (just as in Example 4.2)

$$\frac{h^{(r)}(\eta) - h^{(r)}(0)}{\eta} = \eta^{-1}Q_r(\eta^{-1})\exp(-\eta^{-2}/2) \to 0 \quad \text{as} \quad \eta \to 0+,$$

since $\exp t^2/2 \to \infty$ as $t \to \infty$ faster than any polynomial. Thus $h^{(r)}$ is differentiable at 0 with $h^{(r+1)}(0) = 0$.

Thus if $P(r)$ is true so is $P(r+1)$. But $P(0)$ is trivially true and a simpler version of the inductive step above enables us to deduce $P(1)$ from $P(0)$, so $P(r)$ is true for all r and the stated result is proved.

(iii) Combining (i) and (ii) we have

$$|h^{(r)}(x)| = \left|\sum_{s=0}^{r} a_{r,s}x^{-(r+2s)}\exp(-1/2x^2)\right| \leqslant \sum_{s=0}^{r} |a_{r,s}||x^{-(r+2s)}\exp(-1/2x^2)|$$

$$\leqslant \sum_{s=0}^{r} |a_{r,s}|(r+s)^{(r+2s)/2}$$

$$\leqslant \sum_{s=0}^{r} 4^r r^{r-s}(2r)^{(r+2s)/2} \leqslant \sum_{s=0}^{r} 4^{2r} r^{r-s} r^{(r+2s)/2} = 4^{2r} \sum_{s=0}^{r} r^{3r/2}$$

$$\leqslant 2^{5r} r^{3r/2} \quad \text{for all} \quad x > 0, r \geqslant 1,$$

and so

$$|h^{(r)}(x)| \leqslant 2^{5r} r^{3r/2} \quad \text{for all} \quad x \in \mathbb{R}, r \geqslant 1.$$

■

Lemma 67.2. *With the notation of Lemma* 67.1, *set* $g(x) = h(x-1)h(2-x)$. *Then* $g: \mathbb{R} \to \mathbb{R}$ *is an infinitely differentiable function with*

(a) $g(x) = 0$ *for all* $x \in [1, 2]$,
(b) $g(x) > 0$ *for all* $x \in (1, 2)$,
(c) $|g^{(n)}(x)| \leqslant 2^{6n} n^{3n/2}$ *for all* $x \in \mathbb{R}, n \geqslant 1$.

Proof. Conclusions (a) and (b) are obvious. To check (c) we use Leibnitz's formula

$$(D^n f_1 f_2)(x) = \sum_{r=0}^{n} \binom{n}{r} (D^{n-r} f_1)(x)(D^r f_2)(x)$$

together with Lemma 67.1 to obtain

$$|g^{(n)}(x)| \leqslant \sum_{r=0}^{n} \binom{n}{r} |h^{(n-r)}(x-1)||h^{(r)}(2-x)|$$

$$\leqslant \sum_{r=0}^{n} \binom{n}{r} 2^{5(n-r)}(n-r)^{3(n-r)/2} 2^{5r} r^{3r/2}$$

$$\leqslant 2^{5n} \sum_{r=0}^{n} \binom{n}{r} n^{3(n-r)/2} n^{3r/2} = 2^{5n} \sum_{r=0}^{n} \binom{n}{r} n^{3n/2}$$

$$= 2^{5n} n^{3n/2}(1+1)^n = 2^{6n} n^{3n/2}$$

for all x. ■

The only major addition to the information already obtained in Example 4.2 (and discussed again at the end of Appendix C) are the bounds on the size of the derivatives. We use this information to check that certain series converge.

Lemma 67.3. Let R > 0 *and an integer* $k \geqslant 0$ *be fixed. Then the infinite sums*

$$\sum_{m=0}^{\infty} \frac{g^{(m+k)}(t)}{(2m+1)!} x^{2m+1} \quad \textit{and} \quad \sum_{m=0}^{\infty} \frac{g^{(m+k)}(t)}{(2m)!} x^{2m}$$

converge uniformly for all $|x| \leqslant R, t \in \mathbb{R}$.

Proof. The treatment of the two sums is similar so we shall only deal with

$$\sum_{m=0}^{\infty} \frac{g^{(m+k)}(t)}{(2m)!} x^{2m}.$$

Observe first that

$$(2m)! = \prod_{r=1}^{2m} r \geqslant \prod_{2m \geqslant r \geqslant m/r} ! \geqslant \prod_{2m \geqslant r \geqslant m/8} (m/8) \geqslant 8^{-2m} m^{7m/4} = 2^{-6m} m^{7m/4} \text{ for } m \geqslant 8,$$

and so, using Lemma 67.2,

$$\left|\frac{g^{(m+k)}(t)}{(2m)!}\right| \leqslant \frac{2^{6(m+k)}(m+k)^{3(m+k)/2}}{(2m)!} \leqslant \frac{2^{12m}(2m)^{3(m+k)/2}}{2^{-6m} m^{7m/4}}$$
$$\leqslant 2^{21m} m^{3(m+k)/2 - 7m/4} \leqslant 2^{21m} m^{-m/8} \text{ for all } m \geqslant 12(k+1).$$

Thus

$$\left|\frac{g^{(m+k)}(t)}{(2m)!} x^{2m}\right| \leqslant (2^{21} R^2)^m m^{-m/8} = (2^{21} R^2 m^{-1/8})^m$$
$$\text{for all } m \geqslant 12(k+1),\ |x| \leqslant R \text{ and } t \in \mathbb{R}.$$

Since $m^{-\frac{1}{8}} \to 0$ as $m \to \infty$ it follows that $2^{21} R^2 m^{-\frac{1}{8}} \leqslant 2^{-1}$ for large m and so there exists a constant A with

$$\left|\frac{g^{(m+k)}(t)}{2m!} x^{2m}\right| \leqslant A 2^{-m} \text{ for all } m \geqslant 0, |x| \leqslant R \text{ and } t \in \mathbb{R}.$$

Thus if $M \geqslant N$ and $|x| < R$ we have

$$\left|\sum_{m=N}^{M} \frac{g^{(m)}(t)}{(2m)!} x^{2m}\right| \leqslant A \sum_{m=N}^{M} 2^{-m} \leqslant A 2^{-N+1} \to 0 \text{ as } N \to \infty.$$

The principle of uniform convergence now shows that

$$\sum_{m=0}^{\infty} \frac{g^{m}(t)}{(2m)!} x^{2m}$$

converges uniformly for all $|x| \leqslant R$, $t \in \mathbb{R}$. ∎

The preliminaries are now complete and we can give the answer to the problem posed at the beginning of the chapter.

Example 67.4 (Tychonov). *Let g be as in Lemma 67.2 and write*

$$\phi(x,t)=\sum_{m=0}^{\infty}\frac{g^{(m)}(t)}{(2m)!}x^{2m}.$$

Then $\phi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is infinitely differentiable and

(a) $(\partial\phi/\partial t)(x,t)=(\partial^2\phi/\partial x^2)(x,t)$ *for all* $(x,t)\in\mathbb{R}^2$,
(b) $\phi(x,t)=0$ *for all* $t\notin[1,2], x\in\mathbb{R}$,
(c) $\phi(0,t)>0$ *for all* $t\in(1,2)$.

Proof. By Lemma 67.3, ϕ is well defined and by well known results (e.g. Lemma 53.2) we may differentiate the expression

$$\phi(x,t)=\sum_{m=0}^{\infty}\frac{g^{(m)}(t)}{(2m)!}x^{2m}$$

both with respect to x and with respect to t as often as we like to obtain

$$\frac{\partial^u}{\partial x^u}\frac{\partial^v\phi}{\partial t^v}(x,t)=\sum_{2m\geqslant u}\frac{g^{(m+v)}(t)}{(2m-u)!}x^{2m-u}.$$

In particular,

(a) $(\partial\phi/\partial t)(x,t)=\sum_{m=0}^{\infty}(g^{(m+1)}(t)/(2m)!)x^{2m}=(\partial^2\phi/\partial x^2)(x,t)$.
Since $g(t)=0$ for all $t\notin[1,2]$, $g^{(m)}(t)=0$ for all $t\notin[1,2]$ so
(b) $\phi(x,t)=0$ for all $t\notin[1,2]$, $x\in\mathbb{R}$.
Finally, we observe that
(c) $\phi(0,t)=g(t)>0$ for all $t\in(1,2)$. ■

A simple modification shows that the solution for the heat flow in a semi-infinite rod is also not unique.

Example 67.4′. *Let $\psi(x,t)=\sum_{m=0}^{\infty}(g^{(m)}(t)/(2m+1)!)x^{2m+1}$. Then $\psi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is infinitely differentiable and*

(a) $(\partial\psi/\partial t)(x,t)=(\partial^2\psi/\partial x^2)(x,t)$ *for all* $(x,t)\in\mathbb{R}$,
(b) $\psi(x,t)=0$ *for all* $t\notin[1,2]$, $x\in\mathbb{R}$,
(c) $(\partial\psi/\partial x)(0,t)\neq 0$ *and so* $\psi(\ ,t)\neq 0$ *for all* $t\in[1,2]$
(d) $\psi(0,t)=0$ *for all* $t\in\mathbb{R}$.

Proof. Left as a trivial exercise to the reader. ■

(Taking the restriction of ψ to the region $\{(x,t):x\geqslant 0,t\geqslant 0\}$ gives a non-zero solution to the heat equation for a semi-infinite rod whose initial temperature is 0 everywhere and whose end is kept at 0 throughout.)

The new solutions that we have found consist of a great blast of heat from infinity, and a little thought suggests that the possibility of such solutions spring from the fact that the equation used gives no limiting velocity for the propagation of heat. Thus, provided only it is large enough, a very distant disturbance can produce a rapid change in temperature near the origin. We may suspect that the kind of solution described in Example 67.4 cannot occur if we have an equation like the wave equation where disturbances propagate with a finite velocity, and that even with the heat equation we can exclude such solutions by imposing a condition of slow growth at infinity.

To the applied mathematician Example 67.4 is simply an embarrassment reminding her of the defects of a model which allows an unbounded speed of propagation. To the numerical analyst it is just a mild warning that the heat equation may present problems which the wave equation does not. But the pure mathematician looks at it with the same simple pleasure with which a child looks at a rose which has just been produced from the mouth of a respectable uncle by a passing magician.