Musical String Inharmonicity

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Introduction

In general, we perceive simultaneous discrete frequencies to be more pleasing if the frequencies have simple ratio relationships with each other. In fact, in the classical western theory of music, the perfect fifth is the most consonant pitch pair, and in the ideal case the frequency ratio between the fundamentals of the pitches would be three to two. By contrast, a dissonant pitch pair, the minor second, has a frequency ratio between the fundamentals of about sixteen to fifteen.

This study examines an effect called inharmonicity as it applies to a single plucked musical string. When a musical string vibrates, it carries many discrete frequencies called harmonics. In the ideal case, these harmonics would vibrate with frequencies that are perfect integer multiples of the fundamental, the lowest frequency being carried by the string. Inharmonicity, in the case that this study addresses, is the behavior where harmonics are pulled higher than their ideal values, and the effect becomes more pronounced as the order of the harmonic increases. A very inharmonic string may even sound out of tune with itself, which might give the string a harsh sound with undesirable resonances.

The model that is tested in this study was brought into the active literature by NH Fletcher in the 1960's.[1] It places the cause of inharmonicity solely on the stiffness of the string material. The equation from which this model is created is:

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} - ESK^2 \frac{\partial^4 y}{\partial x^4}$$

Where *u* is mass per unit length, T is the string tension, S is the cross-sectional area, K is half of the radius of the string and E is Young's modulus. Young's modulus for a string is a measure of its resistance to changes in length. In other words, it is a measure of the string's stiffness, so it can be said that it is the parameter that is the first cause for inharmonicity. Equation (1) differs from the wave equation for an ideal string only by the addition of the term that involves the fourth derivative of the displacement of the string from its equilibrium position. So, if the string had no stiffness, E would equal zero, there would be no extra term in this modified wave equation, and there would be no inharmonicity in the string.

The solutions to equation (1) carry the same assumptions as do the solutions for the ideal wave equation: The string has uniform linear density, it is under constant tension, and the displacement of the string is small enough that tension for any infinitesimal segment does not vary as it vibrates. The solutions also assume that the ends of the vibrating section of the strings are supported, such as a guitar string at the bridge or the nut, or a piano string. This assumption leads to the mathematical statements that at the ends of the string:

$$y = 0$$

That is the string is stationary on the support, and:

$$\frac{\partial^2 y}{\partial x^2} = 0$$

The consequence of this mathematical statement is that the influence of the stiffness of the string material does not travel over the support. In other words, there is no affective curvature of the string over the support. The string can be considered to begin and end at the supports, and for any given instant the slope of the string on the support is constant.

Solving equation (1) will provide a prediction for the frequencies that will be carried by a stiff string in the form:

$$f_n = n f_1^0 (1 + Bn^2)^{\frac{1}{2}}$$

Where n is the order of the harmonic. is the fundamental frequency of the string if it had no stiffness. This is governed by the equation:

$$f_1^0 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

B is a parameter that contains information about the physical properties of the string.

$$B = \frac{\pi^2 ESK^2}{TL^2}$$

There are some things to notice about the parameter B in the frequency equation. B is the parameter that describes the degree of inharmonicity of a given string. As B is multiplied by the square of the order of the harmonic, its effect is magnified for higher harmonics. Also, the relationship for frequencies carried by an ideal string can be recovered by setting the string's stiffness, E, to zero, which would make B zero, which would leave .

The definition of B also shows that inharmonicity can be lessened by increasing the tension or the length of the string, or it can be magnified by increasing the string's stiffness or diameter. Clearly, a builder of plucked or hammered stringed musical instruments will be forced to balance the different factors that affect inharmonicity. It is not an effect that can be fully eliminated in any practical way.

So, musical string inharmonicity is not merely an academic curiosity. It is an effect that musicians, instrument makers, and other music adjacent professions deal with on a conscious level. This can be seen in the form of instruments. One of the reasons the sound of a grand piano is preferable to a console piano is because the longer strings lessen the inharmonicity. The less affected harmonic frequencies are then more resonant between different pitches, as the frequency relationships will be closer to simpler ratios.

Piano tuners also deal with inharmonicity in a direct way. They will stretch the octaves of the piano, which causes the higher registers to sit a little sharper than would be ideal. Of course, matters of musical perception are complicated, and any statement about what is good or preferable is bound to come with many caveats. Some say that the octave stretching on a piano help make it "livelier." [2]

The struggle with inharmonicity can also be seen in the form and function of guitars. Guitarists will often retune strings to make the overtones resonate in a preferable way in a given key. The G string, which is the thickest unwound string on a standard electric guitar, is notorious for presenting tuning difficulties for the guitarist. Its harmonics can sound harsh, and methods such as detuning the string slightly are employed to lessen the effect. Inharmonicity also helps to explain why bass guitars have much longer necks. Mathematically, the low frequencies on a bass guitar could be achieved by maintaining the short scale length of a standard guitar and increasing the strings' mass per unit length (thicker strings), or by placing them under much less tension. However, doing those two things would increase the inharmonicity of the strings to a degree that would border on unmusical. So, the design of an instrument, in view of inharmonicity, becomes a balancing act between the factors that define it.

Background

The foundations for this research were laid by Rayleigh in his Theory of Sound in the 1870s[3] and further developed by Fletcher in the 1960s.[1,4] The final form of the equations that we use come from Fletcher's *The physics of musical instruments*, but he laid out the math in an earlier paper[1]. In that paper, Fletcher measured the inharmonicity of a string for each note on a Hamilton upright piano. His measurements covered both the plain, monofilament strings that are found in the midrange and upper registers of the piano, as well as the wound strings that are found in the lower registers of the piano. In the paper B values were derived from the dimensions of the strings, as it was impractical to remove strings from the piano to make measurements about their properties. This led to difference in B calculations between wound and plain strings. It was found that there is an extra torque provided by windings, and after all of the aforementioned is accounted for, it was concluded that equation (2) gives a good approximation for the frequencies of the harmonics of each string.

Fletcher also notes that a previous paper of his found that "the excellence of the tone from a piano can not be said to be greater or less as the value of B becomes greater or less. There must be an optimum value of B for each string and this value has not yet been found. It is certainly not B=0, which would mean that all the partials should be harmonic."

In the literature, there is a large body of research examining inharmonicity as it pertains to pianos. Inharmonicity has long been the explanation for the stretched octaves of a tuned piano. To understand why, consider the concept of beats. When two pitches are played on a piano simultaneously, the combined volume will be a fluctuation of the sound intensity. The rate at which the intensity completes a cycle from loud to soft is called a beat. For example, if a 440Hz tone and 442Hz tone are played together, there will result a beat frequency of 2 Hz. While tuning a piano, it is desirable to maintain a particular beat frequency for a certain interval across the range of the instrument. Due to inharmonicity, it is necessary to stretch the octaves to create the desired beat frequencies[5].

There is also research involving the inharmonicity of guitar strings, with some of it centering on the psychoacoustic aspects of the phenomenon. The form of the guitar lends itself to high levels of inharmonicity. The relatively short scale length and the low tension of the strings, relative to an instrument like a piano, are the main reason for a level of inharmonicity great enough that the inharmonicity of a single guitar strings can be perceptible to a listener. Järveläinen[6] created listening experiments for steel and nylon stringed guitars. Real recordings were used to create a parametric model of the guitar tones so that the inharmonicity of the tones could be controlled. Through the listening experiments, a threshold was found for the perceptibility of inharmonicity that was close to typical values found on the guitar. It was also found that the inharmonicity was more or less detectable depending on whether or not the attack transients were cropped out. This type of understanding of inharmonicity has uses for digital instrument sound synthesis, where sound creators may want to be able to consider all the factors that make a certain tone sound realistic.

There is also an inharmonicity related explanation for the phenomenon of wound guitar strings "going dead" after some amount of playing time. When a string "goes dead," it has a dull sound with shorter sustain. Houtsma[7] found that this is due to the increased inharmonicity of a well-used wound guitar string. Houtsma simulated the stretching and releasing that is done by playing a guitar string and found that it caused a mass redistribution that causes greater inharmonicity in the partials. This inharmonicity then makes it harder to tune the string exactly and gives it less self-resonance which creates the dull, quickly-decaying characteristic.

Clearly, musical string inharmonicity is a phenomenon that has its consequences in many different areas of the practice of music, from instrument and sound design, to tuning and performance. It is something that has been grappled with on a practical level by musicians and music-related professionals ever since the creation of stringed instruments whose strings are excited by plucking or striking.

This research seeks to extend the understanding of musical string inharmonicity by examining strings of varying constructions and materials using the modern tools available today. Fletcher's equation is used to provide an expectation of behavior for inharmonic partials on a string, and audio recordings are used to provide the data about the actual behavior of the string to be compared to the expected behavior. This work differs from previous work in two ways. First, we measure the properties of each string directly, such as its dimensions. Each string's Young's modulus and its mass per unit length is also measured instead of derived as in previous work. As stated in the introduction, the Young's modulus, the stiffness of the string material, is the first cause of inharmonicity in a string's behavior, so measuring it directly represents an acknowledgment of this importance.

The second differentiating component to this study is the variety of strings that are measured and examined. We look at monofilament steel guitar strings, that is strings that do not vary in material or geometry across their cross section, and we look at wound steel guitar strings. We also examine the applicability of Fletcher's equation to nylon guitar strings. Nylon does not provide the same type of linear response to stretching that steel does in the range of tensions on a guitar, so it is not clear that inharmonicity will behave the same way for nylon strings. Furthermore, we look at the wound strings of a nylon set of strings. The wound strings on a nylon set are of a totally different construction than wound steel guitar strings. The nylon wound strings have a core made of a thread of nylon filaments, so it is interesting to examine how that affects the inharmonicity.

Methodology

Each string was examined in two ways, the order of which was determined by practical considerations. First, the string's material properties were measured, so that they could be used with equation (4) to predict a value for their B parameter, which is, in effect, a measure of their inharmonicity. Second, audio of the vibrating string was recorded. With the audio data, a Fast-Fourier Transform provided the discrete frequencies that were present in the audio sample. These frequencies were then used to perform a chi-square fit using equation (2) to provide an experimental value for the B parameter that could be compared against the value that was predicted from material measurements. The two different values for B and the associated uncertainties for each string are then compared.

As previously noted, the order in which these two methods for obtaining a value for the B parameter was performed was determined by practical considerations. For the nylon strings, it was necessary to use strings that were "settled." In practice, nylon strings will complete many cycles of tensioning and stretching before they reach an equilibrium that allows them to hold a steady pitch for a usable length of time. With this in mind, the nylon strings used in this study were installed on a guitar and retuned daily for 2 weeks to allow them to settle. During this time the strings were not played, other than the plucks required to pitch them back up to the desired operating tension. With the nylon strings installed on the guitar, audio samples were recorded. Then the strings were removed, and the dimensions and Young's modulus were measured.

For the steel strings, the strings' dimensions and Young's modulus were measure first, and then they were installed on a monochord apparatus where audio samples were recorded.

Strings' diameters were measured with a micrometer caliper, and length measurements were done with a meter stick. Uncertainties for these measurements were taken from the tolerances of the measurement devices.

Measuring Young's Modulus

Young's modulus, E, is a measure of a material's resistance to changes in length, so to measure the Young's modulus for a string, it is necessary to measure the force provided after measured changes in length.

A given string was clamped at both ends. At one end the clamp was attached to two Pasco force sensors. At the other end, it was attached to a movable stage. The stage was then moved to put the string under a tension that would be slightly higher than its vibrating tension, around 80N for steel strings and around 60N for nylon strings. As the tension was progressively lessened throughout the measurements by moving the stage to reduce the string length, the tension values would pass through the tension that would be used when recording audio of the string as it vibrated. An example of a plot produced by this procedure is shown in figure 1.

For wound strings, the clamps were only attached at segments of exposed cores, and the area used for the Young's modulus calculation was taken to be the area of the core.

It should be noted here that the measurements for nylon strings had to be done rather quickly, to compensate for the "settling" behavior previous-

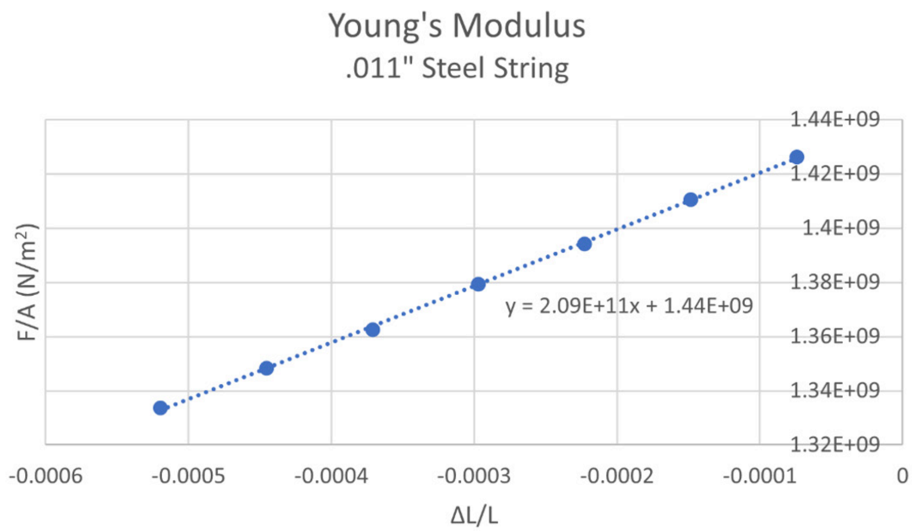

Figure 1 – Scatter plot of the data for the resistance to changes in length for an Ernie Ball .011" steel string. The slope of the trendline of this data gives the Young's modulus.

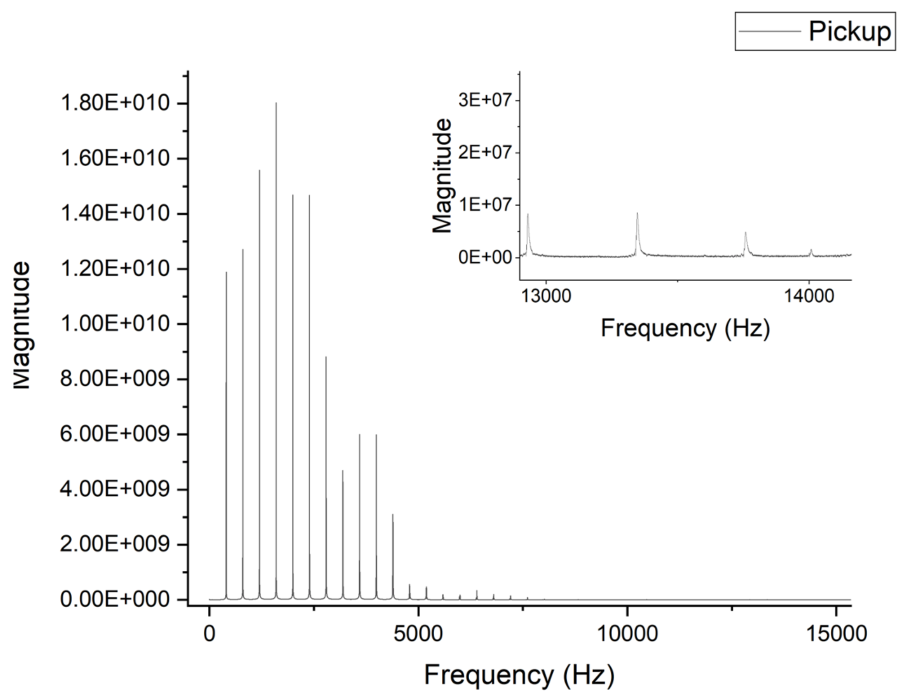

Figure 2 – Plot of output of a FFT from an audio sample of an Ernie Ball .011" steel string. The peaks seen here represent the frequencies present on the string. The inset shows higher harmonic frequencies whose peaks are not visible at the original scaling of the plot.

ly described. When a change is made in the nylon string's length, the tension immediately responds as expected, but it will quickly rebound toward the tension it had before the change in length. Since the minuscule changes in tension while vibrating occur at rates around hundreds of hertz, fast measurements in this procedure would be most relevant to the behavior examined in this study. The values for stiffness obtained are in general agreement with a study done by Lynch-Aird and Woodhouse[8] in which the mechanical properties of nylon strings were examined.

Measurements were plotted and the Young's modulus was given by the slope of a linear chi-square fit. The uncertainty for the Young's modulus was taken from the result of the chi-square.

Audio Samples

Audio samples were mono recordings taken at a sample rate of 44.1 kHz with a sample length of 4-8 seconds. Although longer sample lengths can, in theory, provide more precise outputs for discrete frequencies, the higher harmonics on a musical string die out rather quickly. Since the amplitude for a frequency in the output of a FFT is, loosely speaking, a measure of its presence in the sample, a long sample length will cause short-lasting frequencies to get lost in any noise in the sample.

Audio samples were always recorded by 2 sources concurrently. Microphones, provided by a Zoom H4n handheld recorder and an LG V20, were used for the nylon strings. For the steel strings, a piezo pickup and an electromagnetic pickup wer used.

Strings were each plucked in varying locations. The intensities of the different harmonic frequencies present on a vibrating string are determined largely by the placement of the pluck attack. For example, if a pluck is performed near the end of the string, the higher order harmonics will be emphasized. Varying the location of the pluck attack in the audio samples provided a more complete picture of the string behavior.

Samples from different sources corresponding to the same pluck attack were aligned, and the pluck transients were clipped off the front of the wave forms, then the desired sample length was exported into Origin. The result of this procedure was, for each string, multiple samples from varying sources and varying placements of the pluck attacks. This data was used to aid in the distinguishing of frequencies originating from the string from any other source of noise.

While determining the frequencies present on the string, a single source was chosen to provide the exact frequencies, while the other sources were used as indicators that a given frequency originated from the string and not elsewhere. A plot representative of one used in the above procedure is shown in figure 2. In many spectra, it was possible to clearly see harmonics of the 50^{th} order or greater.

After the string frequencies were determined, they were plotted, and a chi-square fit was performed using equation (2). This provided an experimental value for B, and the uncertainty for this value was taken from the output of the function fit. An example of this function fit is shown in figure 3.

Results and Discussion

The following charts show the results for the monofilament strings that were tested. These were the types of strings for which the model was derived. It should be noted that guitar strings labelled as nickel are typically nickel-plated steel. Both the steel and the nylon strings performed well, which was not completely expected since the microstructures of the materials are so different. It was not obvious that they should behave so similarly when viewed through the lens of this project.

The uncertainties for the values are too small to display in the graph. Generally, they were about 2 orders of magnitude smaller than the principal values. Half of the monofilament strings tested did not provide overlapping value ranges for the B values, but almost all of those only disagreed by a one to two percentage points or less. There is also no strong pattern of the material measurements providing greater or lesser values of the B parameter when compared to the audio-derived B value, or vice versa.

The largest disagreement between the B values provided by the 2 different testing methods occurred with the 22-mil piano steel. There are very plausible lines of speculation for this disagreement. One being the stiffness of the piano wire interfered with the required end conditions of the string. The testing apparatus was not capable

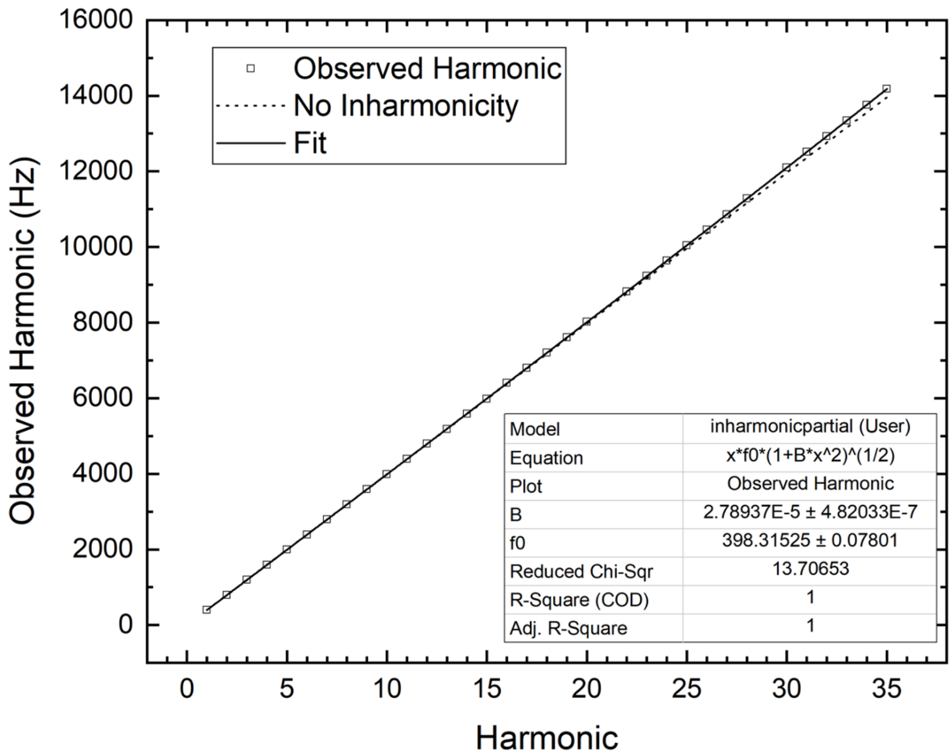

Figure 3 – Plot of the observed harmonic frequencies obtained from Figure 2, along with the fit function's output and the fit line.

String Description	B Measured	ΔB Measured	B Fit	ΔB Fit	Uncertainty Overlap
D'Addario NYXL .011" Steel	2.09E-05	4.74E-07	2.05E-05	1.58E-07	Yes
D'Addario NYXL .014" Steel	5.11E-05	1.14E-06	5.01E-05	3.65E-07	Yes
D'Addario NYXL .018" Steel	1.28E-04	2.64E-06	1.36E-04	1.67E-06	No
DRPureBlues .009" Nickel	9.68E-06	2.35E-07	8.86E-06	4.25E-08	No
DRPureBlues .011" Nickel	2.03E-05	3.80E-07	1.95E-05	8.78E-08	No
DRPureBlues .016" Nickel	8.92E-05	1.64E-06	9.25E-05	1.31E-06	No
Ernie Ball 1009 .009" Nickel	1.14E-05	4.84E-07	1.10E-05	8.22E-08	Yes
Ernie Ball 1011 .011" Nickel	2.87E-05	6.47E-07	2.79E-05	4.82E-07	Yes
Ernie Ball 1016 .016" Nickel	9.53E-05	2.05E-06	9.86E-05	9.91E-07	No
Martin Marquis m1200 .013"	3.54E-05	9.51E-07	3.64E-05	1.57E-07	Yes
Piano Wire .022" Steel	3.54E-04	7.40E-06	3.66E-04	2.67E-06	No
ProArteEJ45 .280" Nylon	3.61E-05	1.29E-06	4.23E-05	2.66E-07	No
ProArteEJ45 .322" Nylon	6.23E-05	3.85E-06	6.38E-05	3.08E-07	Yes
ProArteEJ45 .403" Nylon	1.25E-04	3.11E-06	1.25E-04	7.02E-07	Yes

Table 1 - Results for the monofilament strings.

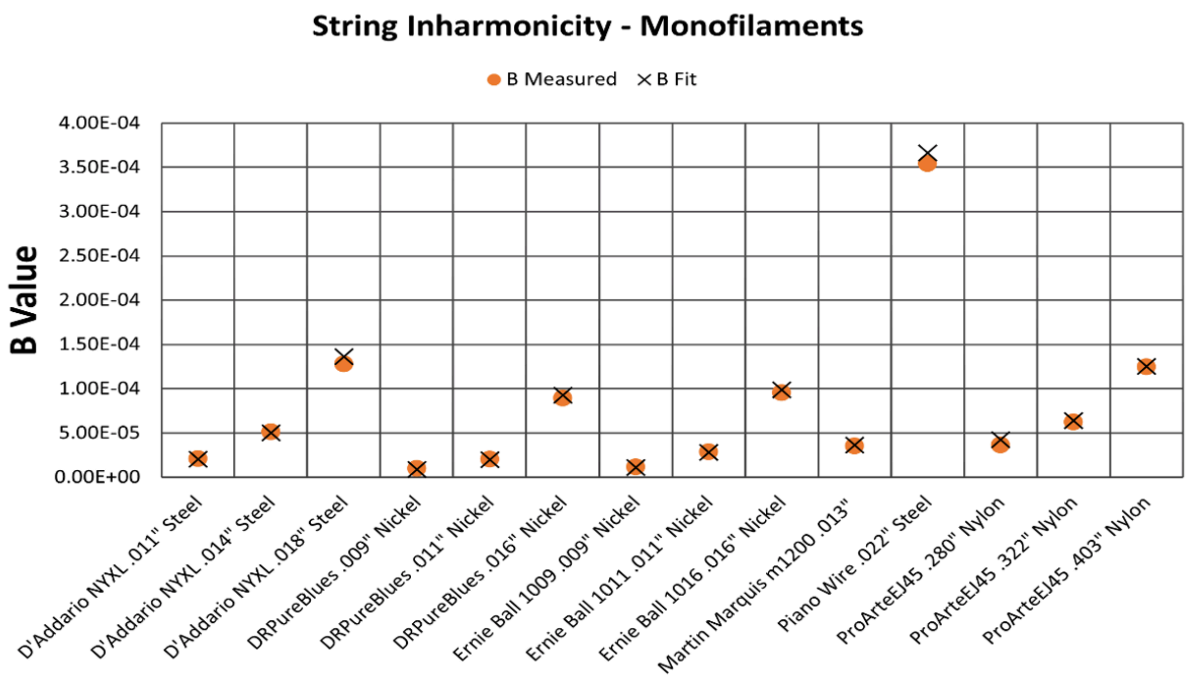

Figure 4 – Scatter plot of the results for the monofilament strings.

String Description	B Measured	ΔB Measured	B Fit	ΔB Fit
D'Addario EXL115 .028" Nickel Wound Hex Core	5.54E-05	6.84E-06	5.87E-05	2.70E-07
D'Addario EXL115 .038" Nickel Wound Hex Core	6.02E-05	4.05E-06	6.88E-05	4.74E-07
D'Addario EXL115 .049" Nickel Wound Hex Core	1.07E-04	1.76E-05	1.34E-04	1.85E-06
D'Addario NYXL .028" Steel Wound Hex Core	4.61E-05	1.67E-06	5.34E-05	1.41E-07
D'Addario NYXL .038" Steel Wound Hex Core	6.89E-05	2.02E-06	8.20E-05	1.27E-06
D'Addario NYXL .049" Steel Wound Hex Core	1.04E-04	1.02E-05	1.51E-04	5.25E-06
DRPureBlues .026" Nickel Wound Round Core	4.45E-05	1.21E-06	4.49E-05	1.84E-07
DRPureBlues .036" Nickel Wound Round Core	7.44E-05	1.94E-06	7.93E-05	7.83E-07
DRPureBlues .046" Nickel Wound Round Core	1.04E-04	2.11E-06	1.31E-04	7.22E-06
ProArteEJ45 .029" Wound Nylon	2.53E-05	5.77E-07	1.84E-05	3.36E-07
ProArteEJ45 .035" Wound Nylon	2.94E-05	8.67E-07	1.42E-05	6.71E-07
ProArteEJ45 .043" Wound Nylon	3.23E-05	7.19E-05	1.99E-05	6.40E-07

Table 2 - Results for wound strings.

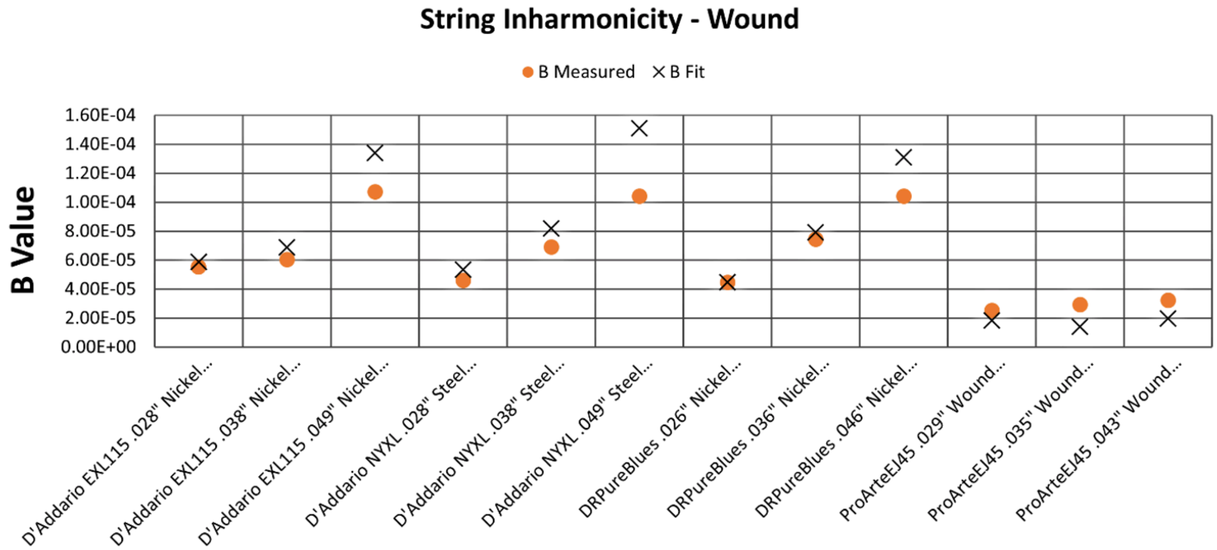

Figure 5 – Scatter plot of the results for wound strings.

of tensioning the string to an amount that would eliminate arcing of the string over the support. It is also possible that for a string of this thickness, the tension was not great enough to keep the string perfectly stationary over the support. This would help explain the behavior of the other thicker steel strings that can be seen in figure 4. During the research, this was an early reminder that string vibration and inharmonicity are phenomena that are quickly complicated by violations of the assumptions that allow the derivation of normal frequencies from the wave equation.

More examples of complicated inharmonicity behavior can be observed in the results for the wound strings.

The geometry and varying composition of wound strings violates the assumptions of the model, but it was interesting to test the behavior. The windings on the strings serve the purpose of increasing the mass per unit length, ideally without affecting the other properties of the string. Increasing the mass per unit length allows a string to support lower frequencies without having to decrease tension or increase the string length. None of the value ranges for the 2 testing methods provided overlaps in the results for the wound strings. In general, the material measurements provided a lower B value than did the frequency measurements. This pattern is reversed in the wound nylon strings. The core of wound nylon strings is loose thin nylon filaments held together by the metal windings of the string. As such, it was exceedingly difficult to measure a valid cross-sectional area that could be used in the calculation for the respective Young's moduli.

The wound string results show the general trend of thicker strings being more inharmonic than thinner strings. The difference in the B values between the 2 testing methods is greater for thicker strings. With the steel strings, the line fit B being consistently greater than the material derived B suggests some systematic behavior. Modern guitar string manufacturing techniques favor using hex shaped cores for the wound strings. It is possible that the sharp edges of the hex shape dig into the windings and the interaction creates an extra restoring force that increases the inharmonicity. This supposition is supported by the round core wound strings offering less difference between their respective B values than the hex core wound strings.

Conclusion

This study demonstrated the effectiveness of attributing the inharmonic behavior of musical strings to the stiffness of the string material. For monofilament strings, the value of the B parameter derived from material measurements was overall in good agreement with the value derived from audio samples. This conclusion holds for both steel and nylon strings. Although some of the uncertainty ranges did not overlap, for monofilament strings the disagreement was within one or two percentage points of the B values. For the wound strings, the disagreement between the B values is not surprising. The model we tested does not account for non-monofilament string constructions. It is likely that there is an interaction between the windings and the core of the string which further complicates the behavior of the string.

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