# Mercer's Theorem and Related Topics[1]

**The Original Result: Mercer's theorem**[2] Let $K = K(t,s)$ be a function defined on $[0,T] \times [0,T]$. Assume that the function $K$ has the following properties:

(1) Continuous:
$$(s,t) \mapsto K(s,t) \text{ is continuous on } [0,T]\times[0,T]; \tag{1.1}$$
(2) Symmetric:
$$K(s,t) = K(t,s), \quad t,s \in [0,T]; \tag{1.2}$$
(3) Positive semi-definite/non-negative definite:
$$\int_0^T\int_0^T K(t,s)f(t)f(s)\,dsdt \geq 0 \text{ for all } \; f \in L_2((0,T)) \tag{1.3}$$

Then
$$K(t,s) = \sum_{m\geq 1} \frac{\varphi_m(t)\varphi_m(s)}{\sigma_m}, \tag{1.4}$$
where
$$\varphi_m(t) = \sigma_m \int_0^T K(t,s)\varphi_m(s)\,ds,\; m\geq 1,\; \int_0^T \varphi_m(t)\varphi_n(t)\,dt = \begin{cases} 1, & m=n,\\ 0, & m\neq n,\end{cases} \tag{1.5}$$
and the series in (1.4) converges absolutely and uniformly on $[0,T]\times[0,T]$:
$$\lim_{M,N\to\infty} \max_{0\leq s,t\leq T} \sum_{m=N+1}^{M} \frac{|\varphi_m(t)\varphi_m(s)|}{\sigma_m} = 0$$
so that
$$\lim_{N\to\infty} \max_{0\leq s,t\leq T} \left| K(t,s) - \sum_{m=1}^{N} \frac{\varphi_m(t)\varphi_m(s)}{\sigma_m} \right| = 0.$$

**Proof.** Recall that $L_2((0,T))$ is a separable Hilbert space with norm $\|f\| = \sqrt{(f,f)_0}$ generated by the inner product $(f,g)_0 = \int_0^T f(t)g(t)\,dt$.

Once we have the appropriate technical set-up that guarantees existence of the functions $\varphi_m$ and numbers $\sigma_m$ in (1.5) (Steps (i), (ii)), it is rather straightforward to establish equality (1.4) in $L_2((0,T)\times(0,T))$, in the sense that the two expressions define the same integral operator (Step (iii).). Proving the equality point-wise (Step (iv)) is more complicated: it requires some *a priori* continuity of the right-hand side of (1.4) and essentially relies on (1.3).

Step (i). Define the operator $\boldsymbol{K} : L_2((0,T)) \to L_2((0,T))$ by
$$\boldsymbol{K} : f(t) \mapsto \boldsymbol{K}[f](t) = \int_0^t K(t,s)f(s)\,ds. \tag{1.6}$$

Direct computations show that the operator $\boldsymbol{K}$ is

- Linear: $\boldsymbol{K}[af+bg] = a\boldsymbol{K}[f] + b\boldsymbol{K}[g],\; a,b\in\mathbb{R},\; f,g\in L_2((0,T))$.
- Bounded: using the Cauchy-Schwarz inequality,
$$|\boldsymbol{K}[f](t)|^2 \leq \left(\int_0^T K^2(t,s)\,ds\right)\cdot\left(\int_0^T f^2(t)\,dt\right),$$
  and then
$$\|\boldsymbol{K}[f]\| \leq C_K\|f\|,\; C_K = \left(\int_0^T\int_0^T K^2(t,s)\,ds\right)^{1/2}. \tag{1.7}$$
- Symmetric (and hence self-adjoint): by (1.2),
$$(\boldsymbol{K}[f],g)_0 = (f,\boldsymbol{K}[g])_0,\; f,g\in L_2((0,T)). \tag{1.8}$$

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[1] Sergey Lotototsky, USC

[2] J. Mercer, *Functions of positive and negative type and their connection with the theory of integral equations*, Philosophical Transactions of the Royal Society, series A, vol. 209, pp. 415–446, 1909. The statement of the theorem is at the bottom of page 444.

- Non-negative: by (1.3),

$$(\boldsymbol{K}f, f)_0 \geq 0,\ f \in L_2((0,T)). \tag{1.9}$$

Step (ii). Slightly more sophisticated analysis shows that the operator $\boldsymbol{K}$ is not just bounded but `compact` (for example, because (1.7) implies that $\boldsymbol{K}$ is `Hilbert-Schmidt`, which is even better than compact). Then the general theory of compact self-adjoint operators implies (1.5):

- The operator $\boldsymbol{K}$ has at most countably many eigenvalues $\lambda_k$, and, by (1.3), $\lambda_k \geq 0$;
- All eigenvalues of $\boldsymbol{K}$ are *non-defective* and all non-zero eigenvalues have finite multiplicity;
- The corresponding eigenfunctions $\varphi_m$, satisfying $\boldsymbol{K}[\varphi_m] = \lambda_m\varphi_m$, form an orthonormal basis in $L_2((0,T))$, so that, for every $f \in L_2((0,T))$,

$$f(s) = \sum_{m=1}^{\infty}(f, \varphi_m)_0\, \varphi_m(s), \tag{1.10}$$

$$\int_0^T |f^2(s)|\, ds = \sum_{m=1}^{\infty}(f, \varphi_m)_0^2, \tag{1.11}$$

and (1.10) is the equality in $L_2((0,T))$.[3]

In particular, the numbers $\sigma_m$ in (1.5) satisfy $\sigma_m = 1/\lambda_m$ for $\lambda_m > 0$; if $\lambda_m = 0$, then the corresponding function $\varphi_m$ does not appear in (1.4).

Step (iii). We can now establish equality (1.4) in $L_2((0,T) \times (0,T))$. To begin, note that, because $K$ is (jointly) continuous, each of the functions $\varphi_m$ corresponding to $\lambda_m > 0$ is continuous. Indeed, using (1.5) and Cauchy-Schwarz,

$$\lim_{h\to 0} |\varphi_m(t+h) - \varphi_m(t)| \leq \sigma_m \lim_{h\to 0} \left( \int_0^T \big(K(t+h, s) - K(t,s)\big)^2\, ds \right)^{1/2} = 0.$$

For $N \geq 1$, consider

$$K_N(t,s) = \sum_{m=1}^{N} \frac{\varphi_m(t)\varphi_m(s)}{\sigma_m}. \tag{1.12}$$

Each function $K_N$ is continuous on $[0,T] \times [0,T]$ and defines the corresponding operator $\boldsymbol{K}_N$ that is positive semi-definite and compact (being of finite rank). It remains to show that

$$\bar{K} = \lim_{N\to\infty} K_N \text{ exists in } L_2((0,T) \times (0,T)). \tag{1.13}$$

Then the corresponding operator $\bar{\boldsymbol{K}}$, being a strong limit of compact operators, is also compact, and the equality $\bar{\boldsymbol{K}} = \boldsymbol{K}$ holds because the operators have the same (non-zero) eigenvalues.

To establish (1.13), note that, because the function $K$ is continuous, we can fix $t$ and consider the corresponding function $f : s \mapsto K(t,s)$ as an element of $L_2((0,T))$. The first equality in (1.5) can now be written as

$$\big(K(t,\cdot), \varphi_m\big)_0 = \frac{\varphi_m(t)}{\sigma_m},$$

so that (1.10) and (1.11) become

$$K(t,s) = \sum_{m\geq 1} \frac{\varphi_m(t)}{\sigma_m}\, \varphi_m(s), \tag{1.14}$$

$$\int_0^T K^2(t,s)\, ds = \sum_{m\geq 1} \frac{\varphi_m^2(t)}{\sigma_m^2} \text{ for every } t \in [0,T]. \tag{1.15}$$

In particular, integrating both sides of (1.15) with respect to $t$ yields

$$\sum_{m\geq 1} \frac{1}{\sigma_m^2} = \int_0^T \int_0^T K^2(t,s)\, dsdt. \tag{1.16}$$

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[3] as opposed to point-wise in $s \in [0,T]$

As a result, for $M > N$,

$$\int_0^T |K_M(t,s) - K_N(t,s)|^2\, ds = \sum_{m=N+1}^{M} \frac{\varphi_m^2(t)}{\sigma_m^2}; \tag{1.17}$$

integrating with respect to $t$ and using (1.16),

$$\lim_{M,N\to\infty} \int_0^T \int_0^T |K_M(t,s) - K_N(t,s)|^2\, dsdt = \lim_{M,N\to\infty} \sum_{m=N+1}^{M} \frac{1}{\sigma_m^2} = 0,$$

and then completeness of $L_2((0,T)\times(0,T))$ implies $\bar{K} = \lim_{N\to\infty} K_N$.

Step (iv). By (1.17), we now have equality (1.14) in $L_2((0,T))$ with respect to $s$ for each fixed $t$, which is still not the same as point-wise in $(t,s)$. Because the function $K$ is continuous and symmetric, we will be able to claim point-wise equality once we show that the right-hand side of (1.14) is a continuous function of $s$ for each fixed $t$.

To begin, notice that condition (1.3) implies $K(t,t) \geq 0$; otherwise, continuity of $K$ will make it possible to violate (1.3) with a suitable choice of the function $f$ that is positive and has support near the point $t_0$ where $K(t_0,t_0) < 0$. On the other hand, by (1.12) and (1.14),

$$\int_0^T \int_0^T \big(K(t,s) - K_N(t,s)\big) f(t) f(s)\, dsdt = \sum_{m>N} \frac{(f,\varphi_m)_0^2}{\sigma_m} \geq 0.$$

In other words, the function $K - K_N$ is continuous and positive semi-definite for each $N$ and therefore $K(t,t) - K_N(t,t) \geq 0$, or

$$\sum_{m=1}^{N} \frac{\varphi_m^2(t)}{\sigma_m} \leq K(t,t) \leq A_K, \text{ where } A_K = \max_{0\leq t\leq T} K(t,t). \tag{1.18}$$

The left-hand side of (1.18) is non-decreasing in $N$ for each $t$ and therefore has a finite limit as $N \to \infty$. As a result, (1.18) implies absolute convergence of the series in (1.14): for every $M > N$, by Cauchy-Schwarz,

$$\begin{aligned} \sum_{m=N+1}^{M} \frac{|\varphi_m(t)\varphi_m(s)|}{\sigma_m} &\leq \left(\sum_{m=N+1}^{M} \frac{\varphi_m^2(t)}{\sigma_m}\right)^{1/2} \cdot \left(\sum_{m=N+1}^{M} \frac{\varphi_m^2(s)}{\sigma_m}\right)^{1/2} \\ &\leq \begin{cases} A_K \left(\sum_{m=N+1}^{M} \frac{\varphi_m^2(t)}{\sigma_m}\right)^{1/2}, & \text{for fixed } t, \\ A_K \left(\sum_{m=N+1}^{M} \frac{\varphi_m^2(s)}{\sigma_m}\right)^{1/2}, & \text{for fixed } s. \end{cases} \end{aligned} \tag{1.19}$$

In fact, the second inequality in (1.19) shows that the convergence in (1.14) is uniform in $s$ for fixed $t$ and uniform in $t$ for fixed $s$. Because each $\varphi_m$ is continuous, we now have continuity of the right-hand side of (1.14) in of $s$ for each fixed $t$, and therefore the *point-wise* equality

$$K(t,s) = \sum_{m\geq 1} \frac{\varphi_m(t)\varphi_m(s)}{\sigma_m}, \ (t,s) \in [0,T]\times[0,T]. \tag{1.20}$$

Next, we recall **Dini's theorem:** If a non-decreasing (or non-increasing) sequence of continuous functions converges point-wise on a closed bounded interval to a continuous function, then the convergence is uniform.

We now note that the function $t \mapsto K(t,t)$ is continuous and, by (1.20), is a point-wise limit of a non-decreasing sequence $\sum_{m=1}^{N} \varphi_m^2(t)/\sigma_m,\ N \geq 1$:

$$\sum_{m\geq 1} \frac{\varphi_m^2(t)}{\sigma_m} = K(t,t). \tag{1.21}$$

Dini's theorem then implies that the convergence in (1.20) is uniform for $t = s$:

$$\lim_{M,N\to\infty} \max_{0\leq t\leq T} \sum_{m=N+1}^{M} \frac{\varphi_m^2(t)}{\sigma_m} = 0. \tag{1.22}$$

When combined with the first inequality in (1.19), equality (1.22) implies uniform convergence of the series in (1.20):

$$\lim_{M,N\to\infty}\ \max_{0\le s,t\le T}\sum_{m=N+1}^{M}\frac{|\varphi_m(t)\varphi_m(s)|}{\sigma_m}=0$$

and therefore

$$\lim_{N\to\infty}\ \max_{0\le s,t\le T}\left|K(t,s)-\sum_{m=1}^{N}\frac{\varphi_m(t)\varphi_m(s)}{\sigma_m}\right|=0,$$

completing the proof of the theorem.

Two immediate **corollaries**: under conditions (1.1)–(1.3),

$$\sum_{m\ge 1}\frac{1}{\sigma_m}=\int_0^T K(t,t)\,dt,\quad \sum_{m\ge 1}\frac{1}{\sigma_m^2}=\int_0^T\int_0^T K^2(t,s)\,dsdt\equiv\int_0^T\left(\int_0^T K(t,s)K(s,t)ds\right)\,dt;$$

the first follows from (1.21) after integrating both sides; the second is (1.16). More generally, if

$$K^{(1)}(t,s)=K(t,s),\ \ K^{(N)}(t,s)=\int_0^T K^{(N-1)}(t,u)K(u,s)\,du,\ \ N>1.$$

then

$$K^{(N)}(t,s)=\sum_{m\ge 1}\frac{\varphi_m(t)\varphi_m(s)}{\sigma_m^N},\tag{1.23}$$

$$\sum_{m\ge 1}\frac{1}{\sigma_m^N}=\int_0^T K^{(N)}(t,t)\,dt,\tag{1.24}$$

and, for each $N\ge 1$, the series in (1.23) converges absolutely and uniformly. Moreover, if $N\ge 2$, then absolute and uniform convergence in (1.23) holds without condition (1.3) [Note that $K^{(2)}$ satisfies (1.3) even if $K$ does not].

## Some related results

**Theorem.** [Hilbert-Schmidt] If $K=K(t,s)$ satisfies (1.1) and (1.2), and

$$f(t)=\int_0^T K(t,s)g(s)\,ds$$

for some $g\in L_2((0,T))$, then

$$f(t)=\sum_{m\ge 1}\frac{(g,\varphi_m)_0}{\sigma_m}\,\varphi_m(t),\ \ t\in[0,T],\tag{1.25}$$

and the series in (1.25) converges absolutely and uniformly.

**Proof.** Once we have (1.4) in $L_2$, equality (1.25) is automatic. Absolute and uniform convergence follow by Cauchy-Schwarz:

$$\left(\sum_{m=N+1}^{M}\left|\frac{(g,\varphi_m)_0}{\sigma_m}\,\varphi_m(t)\right|\right)^2\le\left(\sum_{m=N+1}^{M}(g,\varphi_m)_0^2\right)\cdot\left(\sum_{m=N+1}^{M}\frac{\varphi_m^2(t)}{\sigma_m^2}\right),$$

keeping in mind that $\sum_{m\ge 1}(g,\varphi_m)_0^2\le\|g\|^2$, and, by (1.15), $\sum_{m\ge 1}\varphi_m^2(t)/\sigma_m^2=\int_0^T K^2(t,s)\,ds$.

**Corollary.** If $\lambda\ne\sigma_m$ for all $m$, then, for every continuous function $f=f(t)$, the solution of the integral equation

$$h(t)=\lambda\int_0^T K(t,s)h(s)\,ds+f(t),\ \ t\in[0,T],$$

is unique and is given by

$$h(t)=\lambda\sum_{m\ge 1}\frac{(f,\varphi_m)_0}{\sigma_m-\lambda}+f(t).$$

**Theorem.** [Steklov] Consider the eigenvalue problem for the Sturm-Liuoville operator:

$$\begin{aligned} &-\big(p(x)\varphi'(x)\big)' + q(x)\varphi(x) = \lambda\varphi(x),\ x \in (0,L), \\ &a\varphi(0) - b\varphi'(0) = 0,\ A\varphi(L) + B\varphi'(L) = 0. \end{aligned} \tag{1.26}$$

with $p$ continuously differentiable and strictly positive on $[0, L]$, $q$ continuous and non-negative on $[0, T]$, and non-negative numbers $a, b, A, B$ satisfying $a + b > 0$ and $A + B > 0$. Assume that $\lambda = 0$ is NOT an eigenvalue [which happens if $q$ is not identically equal to zero or if at least one of the numbers $a$ or $A$ is not equal to zero.]

Then

1. there are countably many eigenvalues $\lambda_k$, each eigenvalue is positive and simple, and the corresponding eigenfunctions $\varphi_k$ can be chosen to satisfy $(\varphi_k, \varphi_m)_0 = 0$, $k \neq m$ and $\|\varphi_k\| = 1$;
2. for every function $f$ that is continuously differentiable on $[0, L]$, satisfies the boundary conditions (1.26) and is twice continuously differentiable on $(0, T)$ with $f'' \in L_2((0, T))$, the equality

$$f(x) = \sum_{m=1}^{\infty} (f, \varphi_k)_0\, \varphi_k(x)$$

hods for all $x \in [0, L]$ and the series converges absolutely and uniformly in $[0, L]$.

**Proof.**[4] The result follows from the Hilbert-Schmidt theorem once we show that the corresponding Green's function is continuous and symmetric in $[0.L] \times [0, L]$ and the function $f$ is in the range of a suitable integral operator.

## Generalizations

**The role of conditions** (1.1)–(1.3).

- Condition (1.1) (continuity) and (1.3) (positivity) are used to establish uniform convergence in (1.4); mean-square convergence holds without either continuity or positivity as long as the number $C_K$ from (1.7) is finite, but uniform convergence can fail without (1.3): take a continuous periodic function $F = F(t)$ for which the Fourier series does not converge at every point, and then consider $K(t, s) = F(t - s)$.[5]
- For some purposes, such as the Hilbert-Schmidt theorem, continuity of $K$ can be relaxed to

$$\operatorname{ess\,sup}_t \int_0^T K^2(t, s)\, ds < \infty.$$

- Symmetry is used to ensure that eigenvalues exist and all of the eigenvalues are real.

Extension to complex-valued functions is straightforward: just add complex conjugation $\overline{\phantom{xx}}$ here and there. In particular, (1.2) becomes

$$K(t, s) = \overline{K(s, t)}. \tag{1.27}$$

**Generalization 1:**[6] Let $\mathbb{M}$ be a *compact* finite-dimensional manifold and let $\mu$ be a bounded linear functional on the space $\mathcal{C}(\mathbb{M})$ of continuous complex-valued functions on $\mathbb{M}$, with the extra property that $\mu[f] > 0$ for every $f \in \mathcal{C}(\mathbb{M})$ satisfying $f \geq 0$, $f \not\equiv 0$. If $K = K(x, y)$ is a continuous complex-valued function on $\mathbb{M} \times \mathbb{M}$ satisfying (1.27) and

$$\int_{\mathbb{M}} \int_{\mathbb{M}} K(x, y) f(x) \overline{f(y)} \mu(dx)\mu(dy) \geq 0,\ f \in \mathcal{C}(\mathbb{M}),$$

then

$$K(x, y) = \sum_{m \geq 1} \frac{\varphi_m(x)\overline{\varphi_m(y)}}{\sigma_m}, \tag{1.28}$$

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[4] V. S. Vladimirov, Equations of Mathematical Physics, Marcel Dekker, 1971; page. 277

[5] On the other hand, if a continuous periodic function has non-negative complex Fourier coefficients, then Mercer's theorem implies that the Fourier series converges absolutely and uniformly.

[6] K. Jörgens, Linear Integral Operator, Pitman Books Ltd., 1970; Theorem 8.11.

where

$$\varphi_m(x) = \sigma_m \int_{\mathbb{M}} K(x,y)\varphi_m(y)\,\mu(dy),\ \sigma_m > 0,\ \int_{\mathbb{M}} \varphi_m(x)\overline{\varphi_n(x)}\,\mu(dx) = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases} \tag{1.29}$$

and the series in (1.28) converges absolutely and uniformly.

**Generalization 2:**[7] If

- $\mathfrak{X} = \big(\mathbb{X}, \mathcal{B}(\mathbb{X}), \mu\big)$ is a *finite* positive measure space;
- $K = K(x,y)$ is *bounded* complex-valued function on $\mathbb{X}$ satisfying (1.27) and

$$\int_{\mathbb{X}}\int_{\mathbb{X}} K(x,y)f(x)\overline{f(y)}\mu(dx)\mu(dy) \geq 0,\ f \in L_2(\mathbb{X},\mu), \tag{1.30}$$

then

1. Equality (1.28) holds, and the series converges absolutely and uniformly, $\mu \times \mu$-almost everywhere on $\mathbb{X} \times \mathbb{X}$.
2. The functions $\varphi_m$ from (1.29) are uniformly bounded: $\sup_m \operatorname{ess\,sup}_x |\varphi_m(x)| < \infty$.
3. The series $\sum_{m\geq 1} 1/\sigma_m$ converges.

The part about uniform boundedness of eigenfunctions is interesting (and not at all clear).

**Generalization 3:**[8] If

- $\mathfrak{X} = \big(\mathbb{X}, \mathcal{B}(\mathbb{X}), \mu\big)$ is a measure space, where $\mathbb{X}$ is a "reasonable" topological space (e.g. locally compact metric space), and $\mu$ is a sigma-finite positive measure on the Borel subsets of $\mathbb{X}$ such that $\mu(U) > 0$ for every open set $U \subset \mathbb{X}$;
- $K = K(x,y)$ is a complex-valued continuous function on $\mathbb{X}$ satisfying (1.27) and (1.30);
- $\int_{\mathbb{X}} K(x,x)\mu(dx) < \infty$.

Then equality (1.28) holds, and the series converges absolutely and uniformly *on compact sub-sets of* $\mathbb{X}$.

Note that continuity of $K$ and properties of $\mu$ imply that (1.30) is equivalent to

$$\sum_{k,m=1}^{N} K(x_k, x_m)a_k\overline{a_m} \geq 0$$

for every finite collection of points $x_k \in \mathbb{X}$ and complex numbers $a_k$. With $N = 2$, this, in turn, implies $|K(x,y)|^2 \leq K(x,x)K(y,y)$ and hence square integrability of $K$ over $\mathbb{X} \times \mathbb{X}$: $\iint_{\mathbb{X}\times\mathbb{X}} |K(x,y)|^2\,\mu(dx)\mu(dy) < \infty$.

**Historical comments.** British mathematician JAMES MERCER (1883–1932) was the oldest of nine children; when he was a student at Cambridge (Trinity College), his personal tutor was J. H. Hardy and one of his fellow students was J. E. Littlewood; during the First World War, he was Naval Instructor and participated in the Battle of Jutland, the only major WWI encounter between the British and German fleets. Mercer's summability theorem (in the spirit of the Toeplitz lemma) is also due to him, but is less known.

The German mathematician DAVID HILBERT (1862–1943) requires no introduction.

ERHARD SCHMIDT (1876–1959) was born in Tartu (now Estonia) and was a Ph.D. student of Hilbert; his Ph.D. dissertation (1905) contains what is now known as the Hilbert-Schmidt theorem and other related results; his is also the Schmidt in the Gram-Schmidt othogonalization procedure.

The Russian-Soviet mathematician VLADIMIR ANDREEVICH STEKLOV (1864–1926) got his Ph.D. from Kharkov/Kharkiv University; his advisor was A. M. Lyapunov.

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[7] H. König, Eigenvalue Distribution of Compact Operators, Springer-Basel, 1986; Theorem 3.a.1.

[8] Seems plausible, but reference not available yet