Math 507b, Spring 2021

Homework problems

(1) Let $a_n,\ n \geq 1$, be positive numbers and $b_n = \sum_{k=1}^{n} a_k$. Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n^2} < \infty.$$

(2) For random variables ξ, ξ_n, we say that $\lim_{n\to\infty} \xi_n = \xi$ *completely* if

$$\sum_n \mathbb{P}(|\xi_n - \xi| > \varepsilon) < \infty$$

for every $\varepsilon > 0$. Clarify the connections among point-wise convergence, complete convergence, and almost sure convergence. In particular,

- Is it possible to converge completely but not with probability one?
- Is it possible to converge completely but not point-wise?
- Is it possible to converge point-wise but not completely?
- Is it possible to converge with probability one but not completely?

In each case, either give a proof or construct a counterexample.

(3) Convince yourself that (a) The function

$$f(x) = \frac{x}{1+x},\ x \geq 0,$$

defines the metric on the space of random variables by

$$\rho_f(\xi, \eta) = \mathbb{E} f(|\xi - \eta|).$$

(b) convergence in probability is equivalent to convergence in the metric ρ_f.

(4) Let $\xi_k, k \geq 1$, be Gaussian random variables. Show that the series $\sum_{k\geq 1} \xi_k^2$ converges with probability one if and only if $\sum_{k\geq 1} \mathbb{E}\xi_k^2 < \infty$. Note that ξ_k are not necessarily independent or have zero mean, but you are welcome to start by making these additional assumptions and showing that

$$\sum_k \mathbb{E}\xi_k^2 \leq \left(\mathbb{E}\exp\left(-\sum_k \xi_k^2\right)\right)^{-2}.$$

(5) Assume that $\xi_n,\ n \geq 1$, and ξ are random variables such that $\mathbb{E}|\xi_n| < \infty$ for all n, $\mathbb{E}|\xi| < \infty$, and $\lim_{n\to\infty} \mathbb{E}|\xi_n - \xi| = 0$. Show that $\lim_{n\to\infty} \xi_n = \xi$ in probability and the family $\{\xi_n, n \geq 1\}$ is uniformly integrable.

(6) Generate a sample path of the Poisson process. Try the following two ways: (a) Set up an "exponential clock" and jump every time the clock "rings" (b) Given the time interval, generate the number of events as a Poisson random variable and then generate the times of events using the corresponding number of iid uniform random variables. Try to include the intensity of the Poisson process as a parameter in your procedure. Can you think of any other ways of generating the process?

(7) (a) Generate a random variable having a symmetric α-stable distribution for a given $\alpha \in (0, 2)$. (b) Generate a sample path of a random walk starting at the origin and with increments having a symmetric α-stable distribution. Try one, two, and three dimensions. How many times would you expect the random walk to come close to the origin?

(8) Given a stopping time τ and an adapted sequence $X_n,\ n = 0, 1, 2, \ldots$, confirm that τ and X_τ are $\mathcal{F}_\tau$-measurable.

(9) Let τ and σ be stopping times.

(a) Confirm that $\tau + \sigma,\ \tau \wedge \sigma = \min(\tau, \sigma),\ \tau\sigma$, and $\tau \vee \sigma = \max(\tau, \sigma)$ are stopping times and $\mathcal{F}_{\tau\wedge\sigma} = \mathcal{F}_\tau \bigcap \mathcal{F}_\sigma$.

(b) Confirm that the events $\{\tau = \sigma\}$ and $\{\sigma \le \tau\}$ are $\mathcal{F}_{\tau\wedge\sigma}$-measurable, the event $\{\sigma < \tau\}$ is $\mathcal{F}_\tau$-measurable, and if $\sigma \le \tau$ with probability one, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

(c) Is it possible to express $\mathcal{F}_{\tau\vee\sigma}$ and $\mathcal{F}_{\tau+\sigma}$ in terms of $\mathcal{F}_\tau$ and $\mathcal{F}_\sigma$?

(d) True or false: If $\mathbb{P}(\tau - \sigma \ge 0) = 1$, then $\tau - \sigma$ is a stopping time?

(10) If $S_n, n \ge 1$, is a simple (symmetric) random walk on $\mathbb{Z}^d$, then, for $d = 1, 2, 3$,

$$\lim_{n\to\infty} n^{d/2}\mathbb{P}(S_{2n} = 0) = c_d, \tag{1}$$

and also $c_1 = \pi^{-1/2}$, $c_2 = 1/\pi$. Is equality (1) true for all d? If so, what is the value of c_d?

(11) Let $h = h(t)$, $t > 0$, be a (Borel) measurable real-valued function. Consider the following properties of h:

- LI: h is Lebesgue-integrable on $(0, +\infty)$;
- IRI: the integral $\int_0^{+\infty} h(t)\,dt$ exists as an improper Riemann integral;
- DRI: h is directly Riemann integrable on $(0, +\infty)$.

For each of the following implications, either give a proof that it is true or construct an example illustrating that it is false:

LI⇒IRI; IRI⇒LI; LI⇒DRI; DRI⇒LI; LI⇒IRI; IRI⇒LI.

(12) Prove that a random variable X is arithmetic if and only if the characteristic function $\varphi_X(t) = \mathbb{E}e^{\mathrm{i}tX}$ of X satisfies $|\varphi_X(t_0)| = 1$ for some $t_0 \ne 0$.

(13) Consider a sequence of independent tosses of a fair coin with outcomes H and T.

(a) Compute the probability that HH will appear before HT [It is clearly 1/2].

(b) Compute the expected number of tosses to get HH.

[Here, it is non-trivial, and the answer is 6; if the number we need is x, then $x = (E_H + E_T)/2$, where E_C is the expected number of tosses to get HH if the first toss is C. Then $E_H = 1 + (1 + E_T)/2$, and $E_T = 1 + (E_T + E_H)/2$.]

(c) Compute the expected number of tosses to get HT [The answer is 4.]

(d) Come up with an alternative (qualitative) explanation why the answer in part (b) is bigger than the answer in part (c).

(14) Let $\xi_k, k \ge 1$ be iid random variables with $\mathbb{P}(\xi_k = 0) = \mathbb{P}(\xi_k = 2) = 1/2$. Show that the sequence $X_n = \xi_1 \cdot \ldots \cdot \xi_n$, $n \ge 1$, is a martingale with respect to $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$ [if you want, you can put $X_0 = 1$ and $\mathcal{F}_0 = \{\Omega, \emptyset\}$], but there is no integrable random variable ξ such that $X_n = \mathbb{E}(\xi|\mathcal{F}_n)$.

(15) Let S_n, $n \ge 0$, be a simple symmetric random walk, with $S_0 = 0$. (a) Confirm that $X_n = S_n^2 - n$ is a martingale, and then find an increasing predictable sequence A_n such that $X_n^2 - A_n$ is a martingale. (b) Show that $\mathbb{E}S_\tau = 0$ for every stopping time τ satisfying $\mathbb{E}\sqrt{\tau} < \infty$.

(16) (a) Let S_n, $n \ge 0$, be a random walk (sum of iid random variables ξ_k), with $S_0 = 0$, $\mathbb{E}\xi_k = 0$, and $\mathbb{E}|\xi_k|^r < \infty$ for some r satisfying $1 < r \le 2$. Show that $\mathbb{E}S_\tau = 0$ for every stopping time τ with $\mathbb{E}\tau^{1/r} < \infty$. Give an example illustrating that the result is not true if $r > 2$. (b) Let M_n, $n \ge 0$, be a square-integrable martingale with $M_0 = 0$. Is it true that $\mathbb{E}M_\tau = 0$ for every stopping time τ satisfying $\mathbb{E}\sqrt{\tau} < \infty$?

(17) Let X, and Y be random variables such that, for some sigma-algebra $\mathcal{G}$,

$$\mathbb{E}(X|\mathcal{G}) = Y \quad \text{and} \quad \mathbb{E}(X^2|\mathcal{G}) = Y^2.$$

Show that $\mathbb{P}(X = Y) = 1$.

(18) Let M_n, $n \ge 0$, be a martingale and define $\triangle M_k = M_k - M_{k-1}$.

(a) Show that the sequence

$$\mathcal{E}_n = \frac{e^{M_n}}{\prod_{k=1}^n \mathbb{E}(e^{\triangle M_k}|\mathcal{F}_{k-1})}$$

is a martingale.

(b) Let $\{M_n,\ n \geq 0\}$ be a square-integrable martingale with $M_0 = 0$ and $|\triangle M_k| \leq c$ (for some $c > 0$ and all k and ω). Show that the sequence

$$Z_n = \exp\left(M_n - \frac{\langle M \rangle_n}{2}\right)$$

is a supermartingale. What can happen if we remove the assumption that the jumps $\triangle M_n$ are uniformly bounded?

(19) Let $\{M_n,\ n \geq 0\}$ be a martingale and let τ be a stopping time such that

$$\mathbb{E}|M_\tau| < \infty,\ \mathbb{P}(\tau < \infty) = 1,\ \lim_{n\to\infty} \mathbb{E}\big(|M_n| I(\tau > n)\big) = 0.$$

Show that $\mathbb{E}M_\tau = \mathbb{E}M_0$.

(20) Let $\{X_n,\ n \geq 0\}$ be a positive supermartingale and $\lim_{n\to\infty} \mathbb{E}X_n = 0$. Show that $\lim_{n\to\infty} X_n = 0$ both in L_1 and with probability one.

(21) Let $\xi_k,\ k \geq 1$, be independent and assume that the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \xi_k$$

exists in distribution. Show that the limit also exists with probability one. In other words, if a series of independent random variables converges in distribution, it also converges with probability one. [One possible way to proceed is to use the martingale $e^{\mathrm{i}tS_n}/\mathbb{E}e^{\mathrm{i}tS_n}$ for a suitable t.]

(22) Consider the sequence

$$X_{n+1} = \theta X_n + \xi_{n+1}$$

with unknown θ and independent identically distributed ξ_k having mean zero and finite variance. Confirm that the least-squares estimator of θ based on the observations $X_1, \ldots, X_n$ is strongly consistent as $n \to \infty$, and then try to construct an example of ξ_k when the estimator is not consistent. [You can try Gaussian ξ_k that are not identically distributed, with variance growing fast enough, for example, $\mathbb{E}\xi_k^2 = (k!)^2$].

(23) Here are some other decompositions.

(a) **A generalization of the Doob decomposition.** Let $X = \{X_n,\ n \geq 0\}$ be any adapted sequence with $\mathbb{E}|X_n| < \infty$. Show that we can write $X = M + A$, where M is a martingale and A is predictable; the representation is unique if we assume $A_0 = 0$. [Try $A_n = \sum_{k=1}^{n} \mathbb{E}\big((X_k - X_{k-1})|\mathcal{F}_{k-1}\big)$]

(b) `Krickeberg decomposition.` Let $X = \{X_n,\ n \geq 0\}$ be a submartingale and $\sup_n \mathbb{E}X_n^+ < \infty$. Show that we can write

$$X_n = Y_n - Z_n,$$

where Y is a martingale and Z is a non-negative supermartingale. [Try $Y_n = \lim_{k\to\infty} \mathbb{E}(X_k|\mathcal{F}_n)$]. Is this decomposition of X unique in any sense? An alternative form: every martingale X_n with $\sup_n \mathbb{E}X_n^+ < \infty$ is a difference of two non-negative martingales $M_n^{\pm} = \lim_{k\to\infty} \mathbb{E}(X_k^{\pm}|\mathcal{F}_n)$.

(c) `Riesz decomposition.` Let $X = \{X_n,\ n \geq 0\}$ be a supermartingale with $\inf_n \mathbb{E}X_n > -\infty$. Show that we can write

$$X_n = M_n + Z_n,$$

where M is a martingale and A is a `potential`, that is, a non-negative supermartingale converging to zero, and the representation is unique. [Start with the Doob decomposition of X_n: $X_n = N_n - A_n$, where N is a martingale and A is an increasing predictable process; then argue that $A_\infty = \lim_{n\to\infty} A_n$ exists; then take $M_n = N_n - \mathbb{E}(A_\infty|\mathcal{F}_n)$ and complete the proof.]

(24) (a) Consider a martingale M, a bounded stopping time τ and any other stopping time σ. Then

$$\mathbb{E}(M_\tau|\mathcal{F}_\sigma) = M_{\tau\wedge\sigma}.$$

This is one of the (many) versions of the basic optional stopping theorem.

(b) Consider a martingale M, a stopping time τ, and an $\mathcal{F}_\tau$-measurable random variable η. Show that the sequence N with $N_n = (M_n - M_{n\wedge\tau})\eta$ is a martingale. [Use part (a) to show that $\mathbb{E}N_\sigma = 0$ for every bounded stopping time σ; do not forget to check that N_n is adapted: can replace η with $\eta I(\tau \le n)$].

(25) Let M be a martingale with $\mathbb{E}|M_n|^p < \infty$ for all n and some $p \in (1, +\infty)$. Combine Doob's maximal inequality with Hölder and Fubini to show that

$$\Big(\mathbb{E}\big(\max_{k\le n}|M_k|\big)^p\Big)^{1/p} \le q\Big(\mathbb{E}|M_n|)^q\Big)^{1/q}.$$

Start by writing $M_n^* = \max_{k\le n}|M_k|$ and

$$\mathbb{E}\big(M_n^*\big)^p = (p-1)\int_0^\infty \mathbb{P}(M_n^* > x)x^{p-1}\,dx.$$

(26) Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$ (Borel sigma-algebra), $\mathbb{P}((a,b)) = b - a$ (Lebesque measure); this is sometimes called the Steinhaus probability space.

(a) Let $\mathcal{F}_n$, $n \ge 1$, be the sigma-algebra generated by the intervals

$$(k2^{-n}, (k+1)2^{-n}],\ k = 0, 1, \dots, 2^n - 1.$$

Compute $\mathbb{E}(f|\mathcal{F}_n)$ for a Lebesgue-integrable, Borel-measurable function $f = f(x),\ x \in (0,1)$. The answer is

$$\mathbb{E}(f|\mathcal{F}_n)(x) = \sum_{k=0}^{2^n-1}\left(2^n\int_{k2^{-n}}^{(k+1)2^{-n}} f(y)dy\right) I\big(k2^{-n} < x \le (k+1)2^{-n}\big).$$

(b) Let $f = f(x),\ x \in (0,1)$, be a Lebesgue-integrable, Borel-measurable function. Define $f(x) = 0$ for $x \notin (0,1)$ and let

$$M_f(x) = \sup_{t\in(0,1)} \frac{1}{t}\int_x^{x+t} f(y)\,dy,\ x \in (0,1).$$

Show that, for every $p > 1$,

$$\int_0^1 |M_f(x)|^p\,dx \le \left(\frac{8p}{p-1}\right)^p \int_0^1 |f(x)|^p\,dx.$$

The result is known as Hardy-Littlewood inequality.

(27) `Azuma-Hoeffding Inequality`. If $X = \{X_k,\ k \ge 0\}$, is a martingale with $\mathbb{E}X_k = 0$ and $\mathbb{P}(|X_k - X_{k-1}| \le c_k) = 1$ for some non-random numbers c_k, then, for every $n \ge 1$ and $\lambda > 0$,

$$\mathbb{P}\big(\max_{0\le k\le n}|X_k| > \lambda\big) \le 2\exp\left(-\frac{\lambda^2}{2\sum_{k=1}^n c_k^2}\right).$$

(28) Let $M = \{M_n, n \ge 0\}$ be a martingale with $M_0 = 0$. Consider the following properties of M:

UI: The family $\{M_n, n \ge 0\}$ is uniformly integrable;
H: $\mathbb{E}\sup_n|M_n| < \infty$;
UP: $\sup_n \mathbb{E}|M_n|^p < \infty$ for some $p > 1$.
For each of the following implications, either give a proof or construct a counter-example:
UI⇒U; H⇒UI; UI⇒UP; UP⇒UI; H⇒UP; UP⇒H.

[The collection of martingales with property H is (sometimes) called the `Hardy space`; if we think of UI and UP as the corresponding space too, then UP⊂H⊂UI, with all inclusions strict: the Hardy space is the "correct" intermediate space between uniformly integrable martingales and all L_p martingales, $p > 1$.]

(29) Let $\xi_k,\ k \geq 1$, be iid standard Gaussian random variables. Define $S_n = \sum_{k=1}^n \xi_k$, and

$$M_n = \exp\left(S_n - \frac{n}{2}\right).$$

Confirm that $\{M_n,\ n \geq 1\}$ is a martingale, $\lim_{n\to\infty} M_n = 0$ with probability one, and $\lim_{n\to\infty} \mathbb{E}M_n^p = 0$ if and only if $0 < p < 1$.

(30) Confirm that a non-negative local martingale is a super-martingale.

(31) The "basic martingale CLT" is usually stated for triangular arrays with $\mathcal{F}_k^n = \sigma(\xi_{n,j},\ j = 1, \ldots, k_n)$: if, as $n \to \infty$,

$$\sum_{j=1}^{k_n} \mathbb{P}(|\xi_{n,j}| > \varepsilon \big| \mathcal{F}_{j-1}^n) \to 0,\ \varepsilon > 0,$$

$$\sum_{j=1}^{k_n} \mathbb{E}(\xi_{n,j} I(|\xi_{n,j}| \leq 1) \big| \mathcal{F}_{j-1}^n) \to 0,$$

$$\sum_{j=1}^{k_n} \mathrm{Var}(\xi_{n,j} I(|\xi_{n,j}| \leq 1) \big| \mathcal{F}_{j-1}^n) \to 1,$$

all in probability, then, also as $n \to \infty$,

$$\sum_{j=1}^{k_n} \xi_{n,j} \to \mathcal{N}(0,1)$$

in distribution. State the particular case of this result for $\frac{1}{\sqrt{n}} \sum_{k=1}^n \xi_k$ [taking $\xi_{n,k} = \xi_k/\sqrt{n}$] and then confirm that the case of iid ξ_k (zero mean, unit variance) is covered.

(32) Consider a time-homogenous discrete time Markov chain with finitely many states and transition probabilities $p(i,j)$.

(a) True of False: if $\sum_i p(i,j) = 1$, then the chain is ergodic, and the stationary distribution is (discrete) uniform.

(b) True or False: if the chain is ergodic and the stationary distribution is uniform, then $\sum_i p(i,j) = 1$?

In each case, either give a proof [if you think the statement is true] or construct a counterexample.

(33) Consider the simple symmetric random walk on $[0, L]$ with integer L so that

$$p(i, i \pm 1) = \frac{1}{2},\ i = 1, \ldots, L-1,\ p(0,0) = p(0,1) = p(L,L) = p(L, L-1) = \frac{1}{2}.$$

Confirm that the chain is ergodic and the stationary distribution is uniform on $[0, 1, 2, \ldots, L]$. Find some numbers $C > 0$ and $r \in (0,1)$ such that

$$\max_{i,j} |p^{(n)}(i,j) - 1/(L+1)| \leq Cr^n.$$

How do C and r depend on L?

(34) Let $N = N_n,\ n \geq 1$, be a non-trivial branching process and $\mu = \mathbb{E}N_1 > 1$.

(a) Give an example when $\lim_{n\to\infty} N_n/\mu^n = 0$ with probability one;

(b) Given an example when $\lim_{n\to\infty} N_n/\mu^n \not\equiv 0$ and compute the corresponding limit.

(c) Can $\lim_{n\to\infty} N_n/\mu^n \not\equiv 0$ be infinite with positive probability?

(35) (a) Give an example of a sequence that is strictly stationary but not mean-square stationary.

(b) Give an example of a sequence that is mean-square stationary but not strictly stationary.

(36) (a) Show that, for every finite sequence $n_1 \ldots n_k$, with

$$n_1 \in \{1, 2, \ldots, 9\},\ n_\ell \in \{0, 1, 2, \ldots, 9\},\ \ell = 2, \ldots, k,$$

there exists a positive integer N such that the decimal expansion of the number 2^N starts with $n_1 \dots n_k$. [Start by showing that the map $x \mapsto (x + \log_{10} 2) \bmod 1$ is ergodic.] What about 3^N?

(b) Show that the distribution of the first digit of the sequence $\{2^n,\ n \geq 1\}$ follows `Benford's law` (that is, as $n \to \infty$, the proportion of the numbers in the sequence with the first digit equal to k approaches $\log_{10}(1 + k^{-1})$, $k = 1, \dots, 9$). What about the first two digits? What about 3^n?

(c) As a bonus, determine the smallest n such that 2^n starts with 7.

(37) Let $X_n,\ n \geq 1$, be a stationary ergodic sequence, with each X_k taking values in a finite set. Denote by $p_n = p_n(x_1, \dots, x_n)$ the joint distribution of $(X_1, \dots, X_n)$. Show that the limit

$$\lim_{n\to\infty} \frac{1}{n} \ln p_n(X_1, \dots, X_n)$$

exits with probability one and is non-random. [This is one form of the `Shannon-McMillan-Breiman` (ergodic) theorem]. Start with the iid case.

(38) (a) Confirm that a Gaussian sequence is strictly stationary if and only if it is mean-square stationary.

(b) Let $\{X_n,\ n \geq 1\}$ be a stationary Gaussian sequence with $\mathbb{E}X_n = 0$ and $\lim_{n\to\infty} \mathbb{E}X_1 X_n = 0$. Show that the sequence is ergodic.

(39) Let $\xi_k,\ k \geq 1$, be iid standard normal random variables. Confirm that each of the following represents the standard Brownian motion $W = W(t)$ on $[0, T]$:

$$W(t) = \sum_{k=1}^{\infty} \xi_k M_k(t),$$

where $M_k(t) = \int_0^t m_k(s)ds$ and $\{m_k,\ k \geq 1\}$ is an orthonormal basis in $L_2((0, T))$;

$$W(t) = \sqrt{2T} \sum_{k=1}^{\infty} \xi_k \frac{\sin\big((k - (1/2))\pi t/T\big)}{\pi(k - (1/2))},$$

which is the `Karhunen-Loève` representation/expansion of the standard Brownian motion.

Why is usual Fourier series representation of W not as useful?

(40) Let $W = W(t),\ t \in [0, T]$, be a standard Brownian motion.

(a) Confirm that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Big(W(kT/n) - W((k-1)T/n)\Big)^2 = T,$$

both in L_2 and with probability one.

(b) Confirm that the process $M(t) = W^2(t) - t$ is a martingale. Then find a continuous process $A = A(t)$ so that $M^2(t) - A(t)$ is a martingale. How much further can you go?

(41) Let $W = W(t)$ be a standard Wiener process and let τ be a stopping time. Confirm that

$$\frac{1}{3}\mathbb{E}\sqrt{\tau} \leq \mathbb{E}\Big(\sup_{t\leq\tau} |W(t)|\Big) \leq 3\mathbb{E}\sqrt{\tau}.$$

(42) Let $N = N(t)$ be a Poisson process with intensity λ, so that $\mathbb{E}N(t) = \lambda t$. Confirm that $M(t) = N(t) - \lambda t$ and $M^2(t) - \lambda t$ are martingales.

(43) Let T be a positive random variable ($\mathbb{P}(0 < T < \infty) = 1$). Define the process $X = X(t)$ by $X(t) = I(T = t)$. Identify sufficient (and, if possible, necessary) conditions on the distribution of T for each of the following to happen:

(a) The process X has a modification that is identically equal to zero.

(b) The conditions of the Kolmogorov continuity criterion hold.

(c) The process X does not have a modification that is identically zero.

(d) The filtration generated by X is (right-, left-, simply) continuous.

How the answers to (a)–(d) change if T is a stopping time (on a stochastic basis satisfying the usual conditions).

(44) (a) The `Fractional Brownian motion` with the `Hurst parameter` $H \in (0,1)$ is a Gaussian process $B^H = B^H(t)$, $t \geq 0$, with mean zero and covariance

$$\mathbb{E}B^H(t)B^H(s) = \frac{t^{2H} + s^{2H} - |t-s|^{2H}}{2}$$

Confirm that the trajectories of B^H are Hölder continuous of every order less that H, and that $B^{1/2}$ is the standard Brownian motion.

(b) The `Brownian sheet` $W = W(t,x)$, $t, x > 0$, is a zero-mean Gaussian field with covariance $\mathbb{E}W(t,x)W(s,y) = \min(t,s)\min(x,y)$. What can you say about the process $X(t) = W(t,t)$, $t \geq 0$?

What to remember.

(1) Modes of convergence;
(2) Uniform integrability;
(3) Zero-one laws: Kolmogorov, Hewitt-Savage, Blumenthal;
(4) Stopping time;
(5) Two identities (equalities/equations) of Wald;
(6) Recurrence vs transience for (a) random walk; (b) Markov chain;
(7) Reflection principle;
(8) Ballot theorem;
(9) Arcsine laws;
(10) Martingale/submartingale/supermartingale vs harmonic/sub-harmonic/superhrmonic function;
(11) Doob decomposition (Meyer is for continuous time);
(12) Quadratic variation and covariation, both $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ versions (it gets even more interesting in continuous time);
(13) Optional stopping theorem(s);[1]
(14) Burkholder-Davis-Gundy inequality(ies);
(15) Convergence in L_1 and with probability one for (sub)martingales;
(16) LLN(s) and CLT(s) for martingales;
(17) Theorems of Kakutani and Hájek and Feldman (about equivalence/singularity of measures);
(18) Kolmogorov-named equations in connection with Markov processes: Chapman-Kolmogorov, forward Kolmogorov (Fokker-Plank), backward Kolmogorov;
(19) Strong Markov property;
(20) Ergodic Theorems (the more, the better);
(21) Benford's Law;
(22) Stochastic basis with the usual conditions/assumptions;
(23) Brownian motion;
(24) Poisson process;
(25) Continuity criterion of Kolmogorov;
(26) Different ways two continuous-time stochastic processes can be "the same";
(27) Wiener process vs Brownian motion; Lèvy's characterization of the Wiener process;
(28) Lèvy processes;
(29) Skorokhod representation (embedding) for Brownian motion;
(30) Dambis-Dubinis-Schwarz theorem (a general martingale as a time-changed Brownian motion);
(31) Weak convergence of random processes (as processes) and the Donsker invariance principle;

[1]Not to be confused with *optimal* stopping.

(32) Some other "concrete" examples: simple symmetric random walk, branching (Bienaymé-Galton-Watson) process, Polya urn model, M/G/1, M/M/∞ and other queues, Ehrenfest chain, Bernoulli shift.

Reflective questions for discussions.[2]

(1) Take one homework problem you have worked on this semester that you struggled to understand and solve, and explain how (or if...) the struggle itself was valuable.
(2) What mathematical ideas are you curious to know more about as a result of taking this class? Give one example of a question about the material that you would like to explore further, and explain why you consider this question interesting.
(3) What three theorems did you most enjoy from the course, and why?
(4) Formulate a research question related to the course material that you would like to answer.
(5) Reflect on your overall experience in this class by describing an interesting idea that you learned, why it was interesting, and what it tells you about doing or creating mathematics.
(6) Think of one particular proof [of a result related to the topic of this class] and share your ideas about the ways you think the proof should be improved. [The two super-challenges are the section theorem(s) about stopping times and existence of a progressively measurable modification].

[2]Most are not mine, including the wording. Suggestions for improvement will be part of the discussion.