MATH 445 Mid-Term Exam 2
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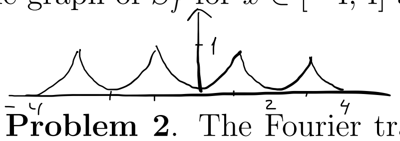
Name: ______________________________

Instructions:

- No notes, no books or other printed materials (including printouts from the web), and no calculators.
- Answer all questions, show your work (when appropriate), upload solutions to Gradescope.
- **There are five problems; each problem is worth 10 points.**

Problem 1. Let $f(x) = x^2$, $|x| < 1$. Denote by $S_f(x)$ the sum of the Fourier series of f. Draw the graph of S_f for $x \in [-4, 4]$ and evaluate (a) $S_f(4)$ (b) $S_f(5/2)$.



$S_f(x) = S_f(x+2) \Rightarrow S_f(4) = S_f(0) = 0$

$S_f(x) = x^2, \quad |x| \le 1. \qquad S_f(5/2) = S_f(1/2) = 1/4.$

Problem 2. The Fourier transform of the function $f(x) = e^{-x^2/2}$ is $\widehat{f}(\omega) = e^{-\omega^2/2}$. Compute the Fourier transform of the function $g(x) = xe^{-x^2}$. Need FT of $h(x) = e^{-x^2}$.

$h(x) = f(\sqrt{2}\,x) \Rightarrow \hat{h}(\omega) = \frac{1}{\sqrt{2}}\hat{f}(x/\sqrt{2}) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}$. $g(x) = x \cdot h(x) \Rightarrow \hat{g}(\omega) = i\hat{h}'(\omega) = (-i\omega/2\sqrt{2})e^{-\omega^2/4}$

Problem 3. Use separation of variables to find a non-constant solution $u = u(t,x)$ of the equation $u_t = u^2 u_x$, defined for $t < 1$ and $x > -1$. $u = f(t)g(x)$ $\quad f'g = f^2g^2\, fg'$

$\frac{f'}{f^3} = gg' = 1 \Rightarrow f = \frac{1}{\sqrt{1-t}}, \quad g = \sqrt{x+1} \Rightarrow u(t,x) = \sqrt{\frac{x+1}{1-t}}$

Problem 4. Solve the following initial value problem:

$$\begin{aligned} u_t - 2u_x &= 0,\ u = u(t,x),\ t > 0,\ x \in \mathbb{R}, \\ u(0,x) &= \sin x. \end{aligned}$$

$u(t,x) = \sin(x + 2t)$

Problem 5. Solve the following initial-boundary value problem:

$$\begin{aligned} u_{tt} &= 9u_{xx},\ u = u(t,x),\ t > 0,\ x \in (0,\pi), \\ u(0,x) &= \sin(2x) - 3\sin(5x), \\ u_t(0,x) &= 0, \\ u(t,0) &= 0, \\ u(t,\pi) &= 0. \end{aligned}$$

$u(t,x) = \sum_k f_k(t)\sin(kx)$

$f_k'' = -9k^2 f_k \Rightarrow f_k(t) = f_k(0)\cos(3kt)$.

$f_k'(0) = 0$

$f_2(0) = 1$

$f_5(0) = -3$

$f_k(0) = 0$ otherwise

$\boxed{u(t,x) = \cos 6t \sin 2x - 3\cos 15t \sin 5x} \Leftarrow$

Properties of the Fourier series and transform

Series	Name	Transform
$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}\,dx$	Forward	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-\mathrm{i}x\omega}\,dx$
$S_f(x) = \sum_{k=-\infty}^{+\infty} c_k(f)e^{\mathrm{i}kx}$	Inverse	$I_f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\omega)e^{-\mathrm{i}x\omega}\,d\omega$
$c_0(f) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)\,dx$	Obvious	$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,dx$
$\sum_{k=-\infty}^{+\infty} c_k(f) = S_f(0) = \frac{\tilde{f}(0+)+\tilde{f}(0-)}{2}$	Obvious	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\omega)\,d\omega = I_f(0)$ $= \frac{f(0+)+f(0-)}{2}$
$\lim_{\|k\|\to\infty} \|c_k(f)\| = 0$	Riemann-Lebesgue: $f \in L_1$	$\lim_{\|\omega\|\to\infty} \|\hat{f}(\omega)\| = 0,\ \hat{f}$ continuous
$\sum_{k=-\infty}^{+\infty} \|c_k(f)\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(x)\|^2\,dx$	Parseval/Plancherel: $f \in L_2$	$\int_{-\infty}^{+\infty} \|\hat{f}(\omega)\|^2\,d\omega = \int_{-\infty}^{+\infty} \|f(x)\|^2\,dx$

Further properties of the Fourier transform

Function	Fourier transform	Function	Fourier transform
$f(x)$	$\hat{f}(\omega) = \mathcal{F}[f](\omega)$	$\hat{f}(x)$	$f(-\omega)$
$f(x-a)$	$e^{-\mathrm{i}a\omega}\hat{f}(\omega)$	$e^{\mathrm{i}ax}f(x)$	$\hat{f}(\omega-a)$
$f(x/\sigma)$	$\sigma\hat{f}(\sigma\omega)$	$e^{-x^2/2}$	$e^{-\omega^2/2}$
$f'(x)$	$\mathrm{i}\omega\hat{f}(\omega)$	$xf(x)$	$\mathrm{i}\hat{f}'(\omega)$
$f''(x)$	$-\omega^2\hat{f}(\omega)$	$x^2f(x)$	$-\hat{f}''(x)$
$\int f(x)\,dx$	$\frac{\hat{f}(\omega)}{\mathrm{i}\omega}$	$\frac{f(x)}{x}$	$\frac{1}{\mathrm{i}}\int \hat{f}(\omega)\,d\omega$
$(f*g)(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$	$f(x)g(x)$	$\frac{1}{\sqrt{2\pi}}(\hat{f}*\hat{g})(\omega)$
$e^{-\|x\|}$	$\sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$	$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}}\,e^{-\|\omega\|}$
$1(\|x\| \le 1)$	$\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$	$\frac{\sin\omega}{\omega}$	$\sqrt{\frac{\pi}{2}}1(\|x\| \le 1)$
$\delta_a(x)$	$e^{-\mathrm{i}\omega a}/\sqrt{2\pi}$	$\cos(ax)$	$\sqrt{\pi/2}\Big(\delta_a(\omega)+\delta_{-a}(\omega)\Big)$