MATH 445 Final Exam
Wednesday, May 4, 2–4pm

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Name: ______________________________

Instructions:

- No notes, no books or other printed materials (including printouts from the web), and no calculators.
- Answer all questions, show your work (when appropriate), upload solutions to Gradescope.
- **There are nine problems. Each of the problems 1–8 is worth 10 points. Problem 9 is a collection of 10 multiple choice questions**

Problem 1.

(a) (5 pts) Compute the line integral $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y,z) = 2x^2y^3 + z^4$, ∇f is the gradient of f, and C is a straight line segment from the point $(0,0,0)$ to the point $(1,1,1)$.

$f(1,1,1) - f(0,0,0) = 3$

(b) (5 pts) Compute the flux of the vector field $\mathbf{F} = \underset{\downarrow}{(3x+2xy)}\,\hat{\boldsymbol{\imath}} + \underset{\downarrow}{(z^2+y^2)}\,\hat{\boldsymbol{\jmath}} + \underset{\downarrow}{(4+x)z}\,\hat{\boldsymbol{\kappa}}$ out of the sphere $\underbrace{(x+1)^2 + y^2 + (z-1)^2 = 4}_{\partial G}$.

$3+2y \qquad 2y \qquad 4+x$

$\int_G (7+4y+x)\,dV = \mathrm{Vol}(G)\,(7+4\bar{y}+\bar{x}) = \frac{4\pi}{3}2^3(7+0-1) = 64\pi$

Problem 2.

(b) (5 pts) Compute $\oint_C \frac{e^{2z}-1}{z^2}\,dz$, where C is the circle $|z| = 4$, oriented counterclockwise.

$\to \frac{1+2z-1}{z^2} \to \frac{2}{z} \to 4\pi i$

(c) (5 pts) Compute the Laurent series expansion of the function $f(z) = \frac{z-2}{z-1}$ around the point $z_0 = 1$.

$1 - \frac{1}{z-1}$ or $-\frac{1}{z-1} + 1$

Problem 3. Determine the solution of the equation

$$y''(x) - 2xy'(x) + 6y(x) = 0$$

satisfying $y(0) = 0$, $y'(0) = 12$.

$y = \sum a_n x^n$

$a_{n+2}(n+1)(n+2) - 2n a_n + 6a_n = 0$

$a_{n+2} = \frac{2n-6}{(n+1)(n+2)}$ $\qquad a_0 = 0$, $a_1 = 12$

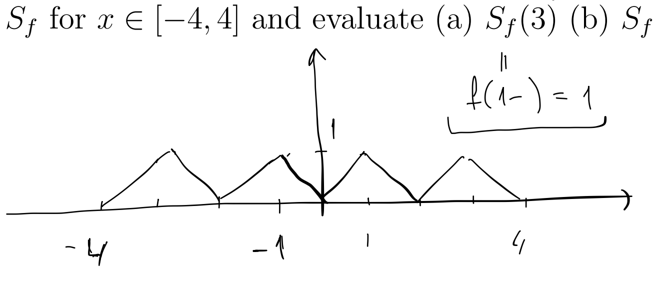
$a_{2k} = 0$

$a_3 = -8$, $a_{2k+1} = 0$, $k > 1$

$y(x) = 12x - 8x^3$

Problem 4. Let $f(x) = |x|$, $|x| < 1$. Denote by $S_f(x)$ the sum of the Fourier series of f. Draw the graph of S_f for $x \in [-4, 4]$ and evaluate (a) $S_f(3)$ (b) $S_f(5/2)$.

$\overset{\|}{f(1-) = 1}$ $\qquad$ $\overset{\|}{f(\tfrac{1}{2}) = \tfrac{1}{2}}$.



Problem 5. The Fourier transform of the function $f(x) = e^{-x^2/2}$ is $\widehat{f}(\omega) = e^{-\omega^2/2}$. Compute the Fourier transform of the function $g(x) = x^2 e^{-x^2/4}$.

$\hat{g}(\omega) = -\sqrt{2}\left(e^{-\omega^2}\right)'' = 2\sqrt{2}\,(1-2\omega^2)\,e^{-\omega^2}$

Problem 6. Use separation of variables to find a non-constant solution $u = u(t,x)$ of the equation $u_t = u^3 u_x$ such that the function $u = u(t,x)$ is defined for all $x > -1$ and $t < 1$.

$u = f(t)\,g(x)$

$f'g = f^3 g^3 f g'$

$\frac{f'}{f^4} = g^2 g' = 1/3.$

$-\frac{1}{3}\left(\frac{1}{f^3}\right)' = \frac{1}{3}\left(g^3\right)' = 1.$

$u(t,x) = \left(\frac{1+x}{1-t}\right)^{1/3}$

Problem 7. Solve the following initial value problem:

$$\begin{aligned} u_t + 2u_x &= 0, \;\; u = u(t,x), \;\; t > 0, \;\; x \in \mathbb{R}, \\ u(0,x) &= e^{\sin x}. \end{aligned}$$

$u(t,x) = u(0, x-2t) = e^{\sin(x-2t)}$

Problem 8. Solve the following initial-boundary value problem:

$$\begin{aligned} u_{tt} &= 4u_{xx}, \;\; u = u(t,x), \;\; t > 0, \;\; x \in (0,\pi), \\ u(0,x) &= 0, \\ u_t(0,x) &= \sin(2x) - 3\sin(5x), \\ u(t,0) &= 0, \\ u(t,\pi) &= 0. \end{aligned}$$

$u = \sum \frac{1}{2k} \sin(2kt)\sin(kx)$

$k=2,\; k=5$

$u(t,x) = \frac{1}{4}\sin(4t)\sin(2x) - \frac{3}{10}\sin(10t)\sin(5x)$

Problem 9. This is a multiple choice part. For each of the 10 questions, circle (or otherwise indicate) the answer you think is correct (there is always only one correct answer). You get two points for each correct selection, zero points for each wrong selection. No need to show your work.

(i) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors. Which ONE of the following expressions is always equal to zero?

$\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{a}$ $\qquad$ $\boldsymbol{a} \cdot \boldsymbol{a}$ $\qquad$ $(\boldsymbol{b} \times \boldsymbol{a}) \times \boldsymbol{b}$ $\qquad$ $\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})$

$a \times b = -\, b \times a$

(ii) Let f be a scalar field and $\boldsymbol{F}$, a vector field. Assuming that all the necessary partial derivatives exist and are continuous, circle the ONE expression that is always equal to zero.

$\boldsymbol{F} \cdot \mathrm{curl}\big(\mathrm{grad}(f)\big)$ $\qquad$ $\mathrm{grad}\big(\mathrm{div}(f\boldsymbol{F})\big)$ $\qquad$ $\mathrm{curl}\big(\mathrm{curl}(f\boldsymbol{F})\big)$ $\qquad$ $\mathrm{grad}\big((\mathrm{grad}\, f) \cdot \boldsymbol{F}\big)$

$\nabla \times (\nabla f) = 0$

z/z^2

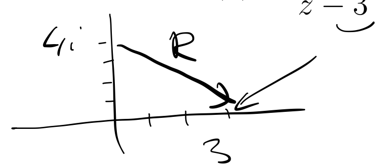
(iii) What is the type of singularity of the function $f(z) = z^{-2}\sin(z)$ at the point $z = 0$?

Simple pole   Pole of order 2   Essential   Not isolated

(iv) What is the radius of convergence of the Taylor series expansion of the function $f(z) = \dfrac{z+5}{z-3}$ around the point $z_0 = 4i$?

$=\sqrt{4^2+3^2}$

2   3   4   5   6



(v) Circle the function $u = u(x, y)$ that is harmonic

$u = x^2 - 2y^2$ $u = e^x \sin(y)$ $u = x^2 + y^2$ $u = xe^y$

(vi) True or false: the Fourier series of the 2π periodic extension of the function $f(x) = x$, $|x| < \pi$, converges uniformly on the interval $[-10, 10]$?

True   False   Need more information

S_f is not continuous.

(vii) Identify the heat equation

$u_t = u_x$ $u_{xx} + u_{yy} = 0$ $u_t = u_{xx}$ $u_{tt} = u_{xx}$ $u_t + uu_x = 0$

(viii) Identify the sequence that converges uniformly to zero on the interval $x \in [0, 1]$:

x^n $\sin(x/n)$ $nx/(1 + nx)$ $nx^2/(n + x)$

$|\sin(x/n)| \leq 1/n$.

(ix) Identify the function $f = f(x)$ for which the Fourier transform $\hat{f}$ is real [that is, the imaginary part of $\hat{f}$ is identically equal to zero.]

$e^{-x^2+x\sin(2x)}$ $e^{-|x|}\sin(2x)$ $e^{-x^4+x^3}$ $e^{-x^2}\big(\cos(2x) + \sin(x)\big)$

the only even function

(x) Let f be a scalar field for which all partial derivatives exist and are continuous. Which of the following expressions defines the Laplacian $\boldsymbol{\Delta} f$ of f?

curl $\big(\text{grad}(f)\big)$ div $\big(\text{grad}(f)\big)$ grad $\big(\text{div}(f)\big)$ grad $\big(|\text{grad}\, f|^2\big)$

$\|$

$(f_x)_x + (f_y)_y + (f_z)_z$

Properties of the Fourier series and transform

Series	Name	Transform
$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}\,dx$	Forward	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-\mathrm{i}x\omega}\,dx$
$S_f(x) = \sum\limits_{k=-\infty}^{+\infty} c_k(f)e^{\mathrm{i}kx}$	Inverse	$I_f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} \hat{f}(\omega)e^{\mathrm{i}x\omega}\,d\omega$
$c_0(f) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)\,dx$	Obvious	$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,dx$
$\sum\limits_{k=-\infty}^{+\infty} c_k(f) = S_f(0) = \frac{\tilde{f}(0+)+\tilde{f}(0-)}{2}$	Obvious	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\omega)\,d\omega = I_f(0)$ $= \frac{f(0+)+f(0-)}{2}$
$\lim\limits_{\lvert k\rvert\to\infty} \lvert c_k(f)\rvert = 0$	Riemann-Lebesgue: $f \in L_1$	$\lim\limits_{\lvert\omega\rvert\to\infty} \lvert\hat{f}(\omega)\rvert = 0,\ \hat{f}$ continuous
$\sum\limits_{k=-\infty}^{+\infty} \lvert c_k(f)\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert f(x)\rvert^2\,dx$	Parseval/Plancherel: $f \in L_2$	$\int\limits_{-\infty}^{+\infty} \lvert\hat{f}(\omega)\rvert^2\,d\omega = \int\limits_{-\infty}^{+\infty} \lvert f(x)\rvert^2\,dx$

Further properties of the Fourier transform

Function	Fourier transform	Function	Fourier transform
$f(x)$	$\hat{f}(\omega) = \mathcal{F}[f](\omega)$	$\hat{f}(x)$	$f(-\omega)$
$f(x-a)$	$e^{-\mathrm{i}a\omega}\hat{f}(\omega)$	$e^{\mathrm{i}ax}f(x)$	$\hat{f}(\omega-a)$
$f(x/\sigma)$	$\sigma\hat{f}(\sigma\omega)$	$e^{-x^2/2}$	$e^{-\omega^2/2}$
$f'(x)$	$\mathrm{i}\omega\hat{f}(\omega)$	$xf(x)$	$\mathrm{i}\hat{f}'(\omega)$
$f''(x)$	$-\omega^2\hat{f}(\omega)$	$x^2f(x)$	$-\hat{f}''(x)$
$\int f(x)\,dx$	$\frac{\hat{f}(\omega)}{\mathrm{i}\omega}$	$\frac{f(x)}{x}$	$\frac{1}{\mathrm{i}}\int \hat{f}(\omega)\,d\omega$
$(f*g)(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$	$f(x)g(x)$	$\frac{1}{\sqrt{2\pi}}(\hat{f}*\hat{g})(\omega)$
$e^{-\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$	$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}}\,e^{-\lvert\omega\rvert}$
$1(\lvert x\rvert \le 1)$	$\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$	$\frac{\sin\omega}{\omega}$	$\sqrt{\frac{\pi}{2}}1(\lvert x\rvert \le 1)$
$\delta_a(x)$	$e^{-\mathrm{i}\omega a}/\sqrt{2\pi}$	$\cos(ax)$	$\sqrt{\pi/2}\Big(\delta_a(\omega)+\delta_{-a}(\omega)\Big)$