MATH 445 Mid-Term Exam 2

Instructions:

- No notes, no books or other printed materials (including printouts from the web), and no calculators.
- Answer all questions, show your work, circle your answers.

****

FROM 2013

Problem 1. (a) (5 pts) The Fourier transform of the function $f(x) = e^{-x^2/2}$ is $\widehat{f}(\omega) = e^{-\omega^2/2}$. Compute the Fourier transform of the function $g(x) = e^{-x^2}$.

(b) (10 pts) The Fourier transform of the function $f(t) = e^{-|t|}$ is $\widehat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$. Use the result to evaluate the integrals $\displaystyle\int_0^\infty \frac{\cos(\omega x)}{1+\omega^2} d\omega$ and $\displaystyle\int_0^\infty \frac{1}{(1+\omega^2)^2} d\omega$.

Problem 2. Use separation of variables to find a non-constant solution $u = u(x,t)$ of the equation $u_t + uu_x = 0$.

Problem 3. Solve the following initial value problem:

$$\begin{aligned} u_t - u_x &= 0, \ u = u(x,t), \ x \in \mathbb{R}, \ t > 0, \\ u(x,0) &= \sin x, \end{aligned}$$

and determine the set of points (x,t) where $u(x,t) = 0$.

Problem 4. Solve the following initial-boundary value problem:

$$\begin{aligned} u_t &= u_{xx}, \ u = u(x,t), \ x \in (0,1), \ t > 0, \\ u(x,0) &= \sum_{n \geq 0} \frac{\sin(2\pi(2n+1)x)}{(2n+1)^2}, \\ u(0,t) &= 0, \\ u(1,t) &= 0. \end{aligned}$$

****

FROM 2020

Note: Table summarizing properties of the Fourier transform was included.

Problem 1. Let $f(x) = x^3$, $|x| < 2$. Denote by $S_f(x)$ the sum of the Fourier series of f. Draw the graph of S_f for $x \in [-6,6]$ and evaluate (a) $S_f(4)$ (b) $S_f(5/2)$.

Problem 2. The Fourier transform of the function $f(x) = e^{-x^2/2}$ is $\widehat{f}(\omega) = e^{-\omega^2/2}$. Compute the Fourier transform of the function $g(x) = xe^{-x^2/2}$.

Problem 3. Use separation of variables to find a non-constant solution $u = u(t,x)$ of the equation $u_t = (u_x)^2$.

Problem 4. Solve the following initial value problem:

$$\begin{aligned} u_t + 2u_x &= 0, \ u = u(t,x), \ t > 0, \ x \in \mathbb{R}, \\ u(0,x) &= \cos x, \end{aligned}$$

and determine the set of points (t, x) where $u(t, x) = 0$.

Problem 5. Solve the following initial-boundary value problem:

$$\begin{aligned} u_t &= 2u_{xx}, \ u = u(t,x), \ t > 0, \ x \in (0,\pi), \\ u(0,x) &= \sin(2x) - 3\sin(5x), \\ u(t,0) &= 0, \\ u(t,\pi) &= 0. \end{aligned}$$