MATH 445 Mid-Term Exam 1

Instructions:

- No notes, no books or other printed materials (including printouts from the web), and no calculators.
- Answer all questions, show your work, circle your answers.

****

FROM 2015

Problem 1.

(a) (5 pts) Compute the line integral $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y, z) = x^2 + yz^3$, ∇f is the gradient of f, and C is a straight line segment from the point $(0, 0, 0)$ to the point $(1, 1, 1)$.

(b) (5 pts) Find the flux of the vector field $\mathbf{F} = 2x\,\hat{\boldsymbol{\imath}} - 3y^2\,\hat{\boldsymbol{\jmath}} + 4z\,\hat{\boldsymbol{\kappa}}$ out of the sphere $x^2+y^2+z^2 = 1$.

Problem 2.

(a) (5 pts) Find the real part of the number $\dfrac{2+i}{3-i}$.

(b) (5 pts) Compute $\oint_C \dfrac{z}{e^z+1}dz$, where C is the circle $|z - \pi i| = \pi$, oriented counterclockwise.

(c) (5 pts) Find the Laurent series expansion of the function $f(z) = \dfrac{2z+1}{z-3}$ around the point $z_0 = 3$.

Problem 3.

(a) (5 pts) Let $f(x) = 2x$, $|x| < 1$. Denote by $S_f(x)$ the sum of the Fourier series of f. Draw the graph of S_f and evaluate (a) $S_f(3)$ (b) $S_f(5/2)$.

(b) (5 pts) Given that

$$x = 2\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sin(nx), \ \ -\pi < x < \pi,$$

compute $\sum_{n\geq 1} \frac{1}{n^2}$.

Problem 4. This is a multiple choice part. For each question, circle the answer you think is correct (there is always only one correct answer). You get three points for each correct selection, zero points for each wrong selection. There are no partial credits in this problem, and no need to show your work.

(a) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors. Which ONE of the following expressions is always equal to zero?

$\boldsymbol{a} \times \boldsymbol{b}$ $\qquad$ $\boldsymbol{a} \cdot \boldsymbol{b}$ $\qquad$ $\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b})$ $\qquad$ $\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})$

(b) Let f be a scalar field and $\boldsymbol{F}$, a vector field. Which ONE of the following expressions never makes sense?

div(curl $\boldsymbol{F}$) curl(div $\boldsymbol{F}$) div(grad f) grad(div $\boldsymbol{F}$)

(c) What is the type of singularity of the function $f(z) = \sin(1/z)$ at the point $z = 0$?

Removable Simple pole Pole of order 3 Essential

(d) What is the radius of convergence of the Taylor series expansion of the function $f(z) = \dfrac{z}{z-4}$ around the point $z_0 = 2$?

1 2 3 4

(e) True or false: the Fourier series of the 2π periodic extension of the function $f(x) = x$, $|x| < \pi$, converges uniformly on the interval $[-10, 10]$?

True False Need more information

****

FROM 2020

Problem 1.

(a) (5 pts) Compute the line integral $\displaystyle\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y, z) = 2x^2y + 3y^2z^3$, ∇f is the gradient of f, and C is a straight line segment from the point $(0, 0, 0)$ to the point $(1, 1, 1)$.

(b) (5 pts) Compute the flux of the vector field $\mathbf{F} = (3x + 2xy)\,\hat{\boldsymbol{\imath}} + (z^2 - y^2)\,\hat{\boldsymbol{\jmath}} + 4z\,\hat{\boldsymbol{\kappa}}$ out of the sphere $x^2 + y^2 + z^2 = 1$.

Problem 2.

(a) (5 pts) Compute the imaginary part of the number $\dfrac{2-i}{3+i}$.

(b) (5 pts) Compute $\displaystyle\oint_C \frac{e^z - 1}{z^3} dz$, where C is the circle $|z| = 4$, oriented counterclockwise.

(c) (5 pts) Compute the Laurent series expansion of the function $f(z) = \dfrac{z-1}{z-3}$ around the point $z_0 = 3$.

Problem 3. Compute a polynomial solution of the equation

$$(*) \qquad xy''(x) - 2xy'(x) + 4y(x) = 0.$$

In other words, find a function $y = y(x)$ that is a polynomial in x and satisfies equation (*) for all x.

Problem 4. This is a multiple choice part. For each question, circle the answer you think is correct (there is always only one correct answer). You get three points for each correct selection, zero points for each wrong selection. No need to show your work.

(a) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors. Which ONE of the following expressions is always equal to zero?

$\boldsymbol{a} \times \boldsymbol{b}$ $\qquad$ $\boldsymbol{a} \cdot \boldsymbol{b}$ $\qquad$ $\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b})$ $\qquad$ $\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})$

(b) Let f be a scalar field and $\boldsymbol{F}$, a vector field. Assuming that all the necessary partial derivatives exist and are continuous, circle the ONE expression that is always equal to zero.

$\operatorname{curl}\big(\operatorname{curl}(f\boldsymbol{F})\big)$ $\qquad$ $\operatorname{grad}\big(\operatorname{div}(f\boldsymbol{F})\big)$ $\qquad$ $\operatorname{div}\big(\operatorname{curl}(f\boldsymbol{F})\big)$ $\qquad$ $\operatorname{grad}\big((\operatorname{grad} f) \cdot \boldsymbol{F}\big)$

(c) What is the type of singularity of the function $f(z) = z^3 \sin(1/z)$ at the point $z = 0$?

Removable $\qquad$ Simple pole $\qquad$ Pole of order 3 $\qquad$ Essential

(d) What is the radius of convergence of the Taylor series expansion of the function $f(z) = \dfrac{z}{z-4}$ around the point $z_0 = 3i$?

2 $\qquad$ 3 $\qquad$ 4 $\qquad$ 5 $\qquad$ 6