Fall 2022, Mathematics of Physics and Engineering II: Homework problems

Homework 1.

PROBLEM 1. Consider four points in $\mathbb{R}^3$: $P(1,1,1)$, $Q(-1,0,2)$, $R(1,-1,-1)$, $S(1+a,0,1-2a)$, where a is a real number.

(1) Compute the coordinates of $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$
(2) Compute the value of a so that the angle $\angle RPS$ is the right angle.
(3) Compute the area of the triangle PQR.
(4) Compute the equation of the plane containing points P, Q, R.
(5) Compute the volume of the parallelepiped spanned by the vectors $\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{PS}$ (it will be a function of a).
(6) For what value of a will the point S lie in the same plane as the points P, Q, R?
(7) Write the vector parametric equation of the line that passes through the point $(1,0,1)$ and is perpendicular to the plane through points P, Q, R, and compute the coordinates of the point of intersection of the line and the plane.

PROBLEM 2. A particle moves so that its position at time t is given by the vector function

$$\mathbf{r}(t) = \langle 1-t^2, t^3, 1+t^2 \rangle, \quad t \geq 0.$$

Compute:

(1) Coordinates of the particle at time $t=1$:
(2) Velocity of the particle for $t \geq 0$:
(3) Speed of the particle for $t \geq 0$:
(4) Acceleration of the particle for $t \geq 0$:
(5) Vector parametric equation of the tangent line to the trajectory at $(0,1,2)$.
(6) The coordinates of the point of intersection of the trajectory with the plane $z-x=2$.
(7) Distance traveled (arc length) from $t=0$ to $t=1$.

Homework 2.

PROBLEM 1. Consider a function $f(x,y) = 2x^2 - xy + y^2 - x + y - 1$.

(1) Compute the gradient of f.
(2) Compute the rate of change of f at $(1,1)$ in the direction toward the origin. Is the function increasing or decreasing in that direction?
(3) Determine the direction of most rapid decrease of f at $(1,1)$ and compute the rate of change of the function in that direction.
(4) (Warning: this one can be time-consuming) Write the equation of the path of steepest ascent on the surface $z = f(x,y)$ starting from point $(0,0,1)$. What path will you get on the topographic map?

PROBLEM 2. Evaluate the following integrals:

(1) $\displaystyle\int_C (2y^2+2xz)dx + 4xydy + x^2dz$, where C is the path $x(t) = \cos t, y(t) = \sin t, z(t) = t, \quad 0 \leq t \leq 2\pi$.
(2) $\displaystyle\oint_C y^2dx + x^2dy$, where C is the boundary of the rectangle with vertices $(1,0)$, $(3,0)$, $(3,2)$, $(1,2)$, oriented counterclockwise.
(3) $\displaystyle\oint_C ydx - zdy + ydz$, where C is the ellipse $x^2+y^2=1; 3x+4y+z=12$ oriented counterclockwise as seen from the point $(0,0,1000)$.

Problem 3. Compute the following quantities using a suitable integral:

(1) The mass of the curve shaped as a helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $t \in [0, 2\pi]$, if the density at every point is the square of the distance of the point to the origin.
(2) The area between x-axis and the curve $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$, $t \in [0, 2\pi]$.
(3) The average distance to the (x, y) plane of the points on the hemisphere $z = \sqrt{1 - x^2 - y^2}$.
(4) The flux of the vector field $\mathbf{F} = \langle x, y, z \rangle$ through the lateral surface of the cylinder $x^2 + y^2 = 1, z \in [0, 2]$.

Problem 4. Let G be the parallelogram with vertices $(0, 0), (4, 0), (5, 1), (1, 1)$, and f, a continuous function. Write the limits in the iterated integrals below:

$$\iint_G f\, dA = \int \int f dx\; dy = \int \int f dy\; dx = \int \int f r dr\; d\theta = \int \int f d\theta\; r dr.$$

Homework 3.

Problem 1. (a) Write equation of the unit sphere in cylindrical coordinates.

(b) Given a twice continuously differentiable function f, write the expression for the Laplacian $\nabla^2 f$ of f in the spherical coordinates (r, θ, φ), where $x = r \cos\theta \sin\varphi$, $y = r \sin\theta \sin\varphi$, $z = r \cos\varphi$.

Problem 2.

(1) Write the number $\frac{2-\mathfrak{i}}{3\mathfrak{i}-4}$ in the form $x + \mathfrak{i}y$.
(2) Write $-1 - \mathfrak{i}$ in the polar form
(3) Solve the following equation: $z^4 - 2z^2 + 2 = 0$. Write the answer in the polar form.
(4) Compute the anti-derivative $\int e^{-2x} \sin(3x)dx$ without integrating by parts.
(5) Compute all the values of $\sqrt[6]{-\mathfrak{i}}$.
(6) Compute all the values of $\sqrt{1 + \mathfrak{i}\sqrt{3}} + \sqrt{1 - \mathfrak{i}\sqrt{3}}$.
(7) Write $\cos(5x)$ and $\cos(6x)$ as polynomials in $\cos x$. [See also: Chebyshev polynomials]

Problem 3. Let $\alpha = e^{2\mathfrak{i}\pi/5}$.

(a) Let $u = \alpha + \alpha^4$. Verify that $u^2 + u - 1 = 0$ and $u = \alpha + \bar{\alpha}$.

(b) Find an algebraic expression for $Re\alpha = \cos(2\pi/5)$.

(c) As the final prize, use the results to construct a regular pentagon using compass and straight edge.

Homework 4.

Problem 1.

(1) For $f(z) = z^3 - 2z + 1$ find $Re(f)$ and $Im(f)$. Is this function analytic?
(2) For $f(z) = ze^z$ find $Re(f)$ and $Im(f)$. Is this function analytic?
(3) For $f(z) = \cos z$ find $Re(f)$ and $Im(f)$. Is this function analytic?
(4) Suppose that f is analytic everywhere and $Re(f) = 0$. What can you say about $Im(f)$?
(5) Determine the value of $a \in \mathbb{R}$ so that the function $u(x, y) = ax^3 + 3xy^2$ is harmonic, and then compute a conjugate harmonic of u.

Problem 2. Find the image of each of the following sets under the map $f(z) = 1/z$:

(1) $\{z : |z| < 1\}$.
(2) $\{z : Re(z) > 1\}$.
(3) $\{z : 0 < Im(z) < 1\}$.

Homework 5.

PROBLEM 1. Determine the radius of convergence of the following power series:

(1) $\sum_{n\geq 0} n\left(\frac{3-\mathfrak{i}}{3+3\mathfrak{i}}\right)^n z^{2n}$.

(2) $\sum_{n\geq 0}(5+\mathfrak{i}+(-1)^n)^n z^{2n}$.

(3) $\sum_{n\geq 0}\frac{(n!)^2}{(2n)!}z^{3n+1}$.

(4) $\sum_{n\geq 0}\frac{n!}{n^n}z^{2n+1}$.

(5) $\sum_{n\geq 1}\frac{z^{2n+1}(4n)!}{(3n)^{4n}(17+(-1)^n)^n}$.

PROBLEM 2. Write the Taylor series of the given function $f = f(z)$ at the given point z_0 and determine the radius of convergence of the series.

(1) $f(z) = \frac{1}{2z+3}$, $z_0 = 0$.

(2) $f(z) = \frac{z^2+1}{z+1}$, $z_0 = 1$.

(3) $f(z) = \frac{1}{(2z-1)^2}$, $z_0 = 0$.

(4) $f(z) = \frac{1}{z^2+2z+2}$, $z_0 = 0$.

PROBLEM 3. Write the Laurent series of the function $f(z) = \frac{1}{2z-z^2}$ in each of the following regions:

(1) $0 < |z| < 2$
(2) $|z-2| > 2$
(3) $0 < |z-2| < 2$
(4) $1 < |z+1| < 3$

PROBLEM 4. Determine the type of singularity at the origin (that is, at the point $z_0 = 0$) of the following functions:

(1) $f(z) = (e^z-1)^2/(1-\cos z)$
(2) $f(z) = (z\cos z - \sin z)/(z\sin z)$
(3) $f(z) = (1-\cos z)/(e^z-1)^4$
(4) $f(z) = \sin(1/z)$
(5) $f(z) = 1/\sin(1/z)$

PROBLEM 5. (I) For each differential equation below, (a) write the general solution as a power series; (b) determine if the equation has a solution that is a polynomial. Everywhere, $y = y(x)$.

(1) $y'' = xy$
(2) $xy'' + (5-x)y' + 2y = 0$
(3) $x^2y'' + xy' + (x^2-9)y = 0$
(4) $y'' - 2xy' + 10y = 0$
(5) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, $n = 0, 1, 2, 3, \ldots$

(II) Given real numbers $\sigma_0, \sigma_1, \sigma_2$, τ_0, τ_1, and λ, confirm that the ordinary differential equation

$$(\sigma_0 + \sigma_1 x + \sigma_2 x^2)y''(x) + (\tau_0 + \tau_1 x)y'(x) = \lambda y(x)$$

has a polynomial solution if [and only if?]

$$\lambda = n\tau_1 + n(n-1)\sigma_2$$

for some non-negative integer n, and then n is the degree of this polynomial.

Homework 6.

Problem 1. Compute the residue $\operatorname*{Res}_{z=z_0} f(z)$ for the following functions f and points z_0:

(1) $f(z) = \dfrac{z+5}{z^3 - z}$, $z_0 = 1$;
(2) $f(z) = \dfrac{z^6 + 3z^4 + 2z - 1}{(z-1)^3}$, $z_0 = 1$;
(3) $f(z) = \dfrac{z}{e^z + 1}$, $z_0 = \pi\mathfrak{i}$;
(4) $f(z) = \dfrac{z}{e^z + 1}$, $z_0 = 3\pi\mathfrak{i}$;
(5) $f(z) = z^2 \sin(1/z)$, $z = 0$.

Problem 2. Evaluate the following integrals (residue method is strongly recommended):

(1) $\displaystyle\oint_{z:|z-2\pi\mathfrak{i}|=2\pi} \frac{z}{e^z+1}\,dz$
(2) $\displaystyle\int_0^{2\pi} \frac{1+2\sin t}{5+4\cos t}\,dt$
(3) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(4+2x+x^2)^3}$
(4) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^n}$, $n = 1, 2, \ldots$, and determine the behavior of the integral as $n \to \infty$.

Problem 3. Compute the power series expansion of the solutions of the following equations:

(1) $w''(z) - zw(z) = 0$, $w(0) = 0$, $w'(0) = 1$;
(2) $z^2 w''(z) + zw'(z) + z^2 w(z) = 0$, $w(0) = 1$, $w'(0) = 0$.
(3) $w''(z) - zw'(z) + 2w(z) = 0$, $w(0) = -1$, $w'(0) = 0$.

Problem 4. Determine the values of the parameter λ for which the following equations have polynomial solutions:

(1) $w''(z) - 2zw'(z) + \lambda w(z) = 0$
(2) $(1-z^2)w''(z) - zw'(z) + \lambda w(z) = 0$
(3) $(1-z^2)w''(z) - 2zw'(z) + \lambda w(z) = 0$
(4) $zw''(z) + (1-z)w'(z) + \lambda w(z) = 0$

Homework 7. Solve each problem. Then make and solve 1-2 similar problems that you think are suitable for a one-hour closed-book in-class exam.

Problem 1. Give an example of a power series that converges at one point on the boundary of the disk of convergence and diverges at another point. Explain why the convergence in this example is necessarily conditional (in other words, explain why the absolute convergence at one point on the boundary implies absolute convergence at all points of the boundary).

Problem 2. For each sequence of functions below, identify the limit and determine whether the convergence is uniform:

(1) x^n, $x \in (0, 1)$

(2) $\sin(x/n)$, $x \in (-\pi, \pi)$
(3) $nx/(1+nx)$, $x \in (0,1)$
(4) $nx^2/(n+x)$, $x \in (0,1)$

Food for thought. Is it possible for a series of functions to converge uniformly but not absolutely?

PROBLEM 3. Determine whether each of the following series converges (a) absolutely for each x in the indicated interval (b) uniformly over the indicated interval.

(1) $\sum_{n\geq 0} nx^n$, $|x| < 1$.
(2) $\sum_{n\geq 1} \frac{\cos(nx)}{n^2}$, $-\infty < x < +\infty$.
(3) $\sum_{n\geq 1} \frac{x^n}{n^{3/2}}$, $|x| < 1$.
(4) $\sum_{n\geq 1} \frac{x^n}{n!}$, $-\infty < x < +\infty$.

PROBLEM 4. For functions f, g below, find a, b, c, d so that $g(x) = a + bf(cx + d)$.

(1) $f(x) = x, 0 \leq x \leq \pi$, f is even and 2π-periodic; $g(x) = 2x, 0 \leq x \leq 1/2; g(x) = 2 - 2x, 1/2 \leq x \leq 1$, g is odd and is periodic with period 2.
(2) $f(x) = 1, 0 \leq x < \pi; f(x) = 0, \pi \leq x < 2\pi$, f is 2π-periodic; $g(x) = 1, 0 \leq x < 1/2; g(x) = -1, 1/2 \leq x < 1$, g is periodic with period 1.
(3) $f(x) = x, -\pi \leq x < \pi$, f is 2π-periodic; $g(x) = x, 0 \leq x < 1$, g is periodic with period 1.

PROBLEM 5. Compute the Fourier series expansion of each of the six functions in problem 4. (You will have to compute some integrals, but not for all six functions).

PROBLEM 6. The function $f(x) = x^2, 0 < x < 1$ is to be expanded in a Fourier series. You have three options:
(a) take the periodic extension of the function with period 1.
(b) take the periodic extension of the even extension of the function with period 2.
(c) take the periodic extension of the odd extension of the function with period 2.
Draw the pictures of the sum of the resulting Fourier series in each case. Which option would you choose and why? (Note: to answer these questions you do not need to compute the Fourier coefficients of the function.)

PROBLEM 7. Use the results of Problem 5 to evaluate the following infinite sums:

(1) $\sum_{k\geq 0} \frac{(-1)^k}{2k+1}$
(2) $\sum_{k\geq 0} \frac{1}{(2k+1)^4}$
(3) $\sum_{k\geq 0} \frac{1}{(2k+1)^2}$
(4) $\sum_{k\geq 1} \frac{1}{k^2}$

PROBLEM 8. Let $f(x) = 2x$, $|x| < 1$. Denote by $S_f(x)$ the sum of the Fourier series of f. Draw the graph of S_f and evaluate (a) $S_f(3)$ (b) $S_f(5/2)$.

Homework 8. Solve each problem. Then make and solve 1-2 similar problems that you think are suitable for a one-hour closed-book in-class exam.

PROBLEM 1. Compute the Fourier transform of the function f in each of the following cases:

(1) $f(x) = e^{-2x}, x > 0$, $f(x) = 0$ otherwise.

(2) $f(x) = x, a < x < b$, $f(x) = 0$ otherwise.

(3) $f''(x) - f(x) = u(x)$, where $u(x) = 1, |x| < 1$, $u(x) = 0$ otherwise.

Problem 2. Compute the Fourier transform of the function $f(x) = e^{-|x|}$ and use the result to evaluate the integrals $\displaystyle\int_0^\infty \frac{\cos(wx)}{1+w^2}dw$ and $\displaystyle\int_0^\infty \frac{1}{(1+w^2)^2}dw$.

Problem 3. (a) The Fourier transform of the function $f(x) = e^{-x^2/2}$ is $\widehat{f}(\omega) = e^{-\omega^2/2}$. Compute the Fourier transform of the function $g(x) = e^{-ax^2}$, $a > 0$.

(b) In the previous problem you learned that the Fourier transform of the function $f(t) = e^{-|t|}$ is $\widehat{f}(\omega) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{1+\omega^2}$. Use the result to compute the Fourier transform of the following functions (i) $f(t) = 1/(1+t^2)$; (ii) $f(t) = a/(b+ct^2)$, $a, b, c > 0$.

Problem 4. Confirm that if

$$h(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy,$$

then $\hat{h}(w) = \sqrt{2\pi}\hat{f}(w)\hat{g}(w)$.

Problem 5. Let $f = f(x)$ be a bounded continuous function such that $\int_{-\infty}^{+\infty} |f(x)|dx < \infty$ and $\int_{-\infty}^{+\infty} f^2(x)dx = 1$. For $t > 0$ define

$$u(t,x) = \frac{1}{\sqrt{4\pi \mathfrak{i} t}} \int_{-\infty}^{\infty} e^{\mathfrak{i}(x-y)^2/(4t)} f(y)dy, \quad \mathfrak{i} = \sqrt{-1}.$$

Verify that, for all $t > 0$,

$$\int_{-\infty}^{\infty} |u(t,x)|^2 dx = 1.$$

Food for thought. Is it possible that f is bounded, continuous, $\int_{-\infty}^{+\infty} |f(x)|dx < \infty$, but $\int_{-\infty}^{+\infty} f^2(x)dx = \infty$? Is it possible that f is bounded, continuous, $\int_{-\infty}^{+\infty} f^2(x)dx < \infty$, but $\int_{-\infty}^{+\infty} |f(x)|dx = \infty$? What if we drop the assumption that f is bounded?

Homework 9.

Problem 1. Find the general solution of the equation $u_t + \sin(t)\, u_x = 0$.

Problem 2. Find the general solution of the equation $x^2 u_x - y^2 u_y = 0$.

Problem 3. Compute the eigenvalues and eigenfunctions of the Laplacian (a) in the disk of radius R, with zero Neumann boundary conditions; (b) in the ball of radius R, with zero Dirichlet boundary conditions; (c) in the ball of radius R, with zero Neumann boundary conditions.

Homework 10.

Problem 1. Find a solution of the following equations using separation of variables.

(1) $u_{xx} + u_{yy} = 0$
(2) $y^2 u_x - x^2 u_y = 0$
(3) $u_x + u_y = (x+y)u$
(4) $xu_{xy} + 2yu = 0$
(5) (i) $u_t + au^r u_x = 0,\ a \neq 0,\ r \neq 0$; (ii) $u_t + auu_x = u,\ a \neq 0$
(6) $u_t = (u^\gamma)_{xx},\ \gamma > 0$

PROBLEM 2. Solve the following initial-boundary value problems.

(1) $u_{tt} = 4u_{xx}$, $t > 0$, $0 < x < 1$, $u_x|_{x=0} = u_x|_{x=1} = 0$, $u|_{t=0} = f(x)$, $u_t|_{t=0} = f_1(x)$, where $f(x) = x, 0 \le x \le 1/2$, $f(x) = 1 - x, 1/2 \le x \le 1$; $f_1(x) = \cos(2\pi x)$.
(2) $u_{tt} = u_{xx} + \sin x$, $t > 0$, $0 < x < \pi$, $u|_{t=0} = \sin(2x)$, $u_t|_{t=0} = \sin x$, $u|_{x=0} = u|_{x=\pi} = 0$.
(3) $u_{tt} = 4u_{xx}$, $t > 0$, $0 < x < 1$, $u_x|_{x=0} = u_x|_{x=1} = 0$, $u|_{t=0} = u_t|_{t=0} = f(x)$, where $f(x) = x, 0 \le x \le 1/2$, $f(x) = 1 - x, 1/2 \le x \le 1$.

PROBLEM 3. Suppose that $q = q(x), 0 \le x \le 1$ is a continuous non-negative function and v_1, v_2 are nontrivial solutions of $v_i'' - q(x)v_i = a_i v_i$, $v_i'(0) - v_i(0) = v_i'(1) - v_i(1) = 0$, $i = 1, 2$, $a_1 \ne a_2$. What is the value of $\int_0^1 v_1(x)v_2(x)dx$?

PROBLEM 4. Solve the following initial value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \ u = u(x,t), \ x \in \mathbb{R}, \ t > 0, \\ u(x,0) &= \sin x, \\ u_t(x,0) &= 0 \end{aligned}$$

and determine the set of points (x,t) such that $u(x,t) = 0$.

Homework 11.

PROBLEM 1. Find the general solution of the boundary value problems

(1) $u_t = u_{xx}$, $t > 0$, $0 < x < 1$, $u|_{x=0} = 0$, $u_x|_{x=1} = 0$.
(2) $u_t = u_{xx}$, $t > 0$, $0 < x < 1$, $u|_{x=0} = 0$, $(u - u_x)|_{x=1} = 0$.
(3) $u_t = u_{xx}$, $t > 0$, $0 < x < 1$, $u|_{x=0} = 1$, $u|_{x=1} = 0$.
(4) $u_t = u_{xx} + t^2 u$, $t > 0$, $0 < x < 1$, $u|_{x=0} = 0$, $u|_{x=1} = 0$.

PROBLEM 2. Solve the following boundary value problems

(1) $u_{xx} + 4u_{yy} = 0$, $(x,y) \in G = (0,\pi) \times (0,\pi)$; $u(0,y) = u(x,0) = u(\pi,y) = 0$, $u(x,\pi) = \sin x$
(2) $u_{xx} + u_{yy} = -1$, $(x,y) \in G = (0,\pi) \times (0,\pi)$; $u|_{\partial G} = 0$
(3) $u_{xx} + u_{yy} = -1$, $x^2 + y^2 < 1$; $u(x,y) = 0$, $x^2 + y^2 = 1$
(4) $\nabla^2 u(r,\theta) = 0$, $r < 1$; $u_r|_{r=1} = 1$.

PROBLEM 3. Use the power series representation of the Bessel functions

$$J_N(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+N)!} \left(\frac{z}{2}\right)^{2k+N}, \quad N = 0, 1, 2, 3, \ldots,$$

to verify the following properties of the Bessel functions:

(1) $J_N(-z) = (-1)^N J_N(z)$,
(2) $(z^N J_N(z))' = z^N J_{N-1}(z)$,
(3) $(z^{-N} J_N(z))' = -z^{-N} J_{N+1}(z)$,
(4) $zJ_N'(z) = NJ_N(z) - zJ_{N+1}(z) = -NJ_N(z) + zJ_{N-1}(z)$,
(5) $\int_{\mathcal{C}(0,z)} \zeta^N J_{N-1}(\zeta)d\zeta = z^N J_N(z)$, $\mathcal{C}(0,z)$ is a path from 0 to z.

PROBLEM 4. Consider the initial value problem $u_t = 0.5u_{xx}$, $x \in (-\infty, +\infty), t > 0$, $u|_{t=0} = f(x)$.

(1) Compute u if $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.
(2) Suppose that $0 \le f(x) \le 10$ and $\int_{-\infty}^{+\infty} f(x)dx = 1$. Show that, for all $t > 0$, $0 \le u(x,t) \le 10$ and $\int_{-\infty}^{+\infty} u(x,t)dx = 1$.

Homework 12.

PROBLEM 1. Solve the following initial-boundary value problems in polar coordinates, with $u = u(t, r, \theta)$ and a given nice function $f = f(r)$:

(1) $u_t = \nabla^2 u$, $r < 2$; $u(t, r, \theta) = 0$, $r = 2$, $u(0, r, \theta) = f(r)$.
(2) $u_{tt} = \nabla^2 u$, $r < 1$; $u(t, r, \theta) = 0$, $r = 1$, $u(0, r, \theta) = f(r)\cos\theta$, $u_t(0, r, \theta) = 0$.

PROBLEM 2. Let $f = f(t), t \geq 0$ be a continuous function. Show that the function

$$u(t, x) = \int_0^t f(s) \frac{x}{\sqrt{4\pi c^2 (t-s)^3}}\, e^{-\frac{x^2}{4c^2(t-s)}}\, ds$$

satisfies $u_t = c^2 u_{xx}$, $t > 0, x > 0$, $u|_{t=0} = 0$, $u|_{x=0} = f(t)$.