MATH 445 Final Exam

Instructions:

- No notes, no books or printouts from the web, and no calculators.
- Answer all questions, show your work, circle your answers.

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FROM 2013

Problem 1. (10 pts)

(a) (5 pts) Compute the line integral $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y,z) = x^2 + y^3 z^5$, ∇f is the gradient of f, and C is a straight line segment from the point $(0,0,0)$ to the point $(1,3,2)$.

(b) (5 pts) Let C be the curve $x^2 + y^2 = 1, 2x + y + z = 10$, oriented counterclockwise as seen from the point $(0,0,20)$. Compute the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = \langle x^2 + 2xy, xy, 3yz + x \rangle$. (Do not even think of evaluating the integral directly).

Problem 2. (10 points) Compute the power series expansion of the solution of

$$w''(z) - 2zw'(z) + 4w(z) = 0, w(0) = 1, w'(0) = 0.$$

Problem 3.

(a) (5 pts) Find the Laurent series expansion of the function $f(z) = \dfrac{2z+1}{z-3}$ around the point $z_0 = 3$.

(b) (5 pts) Compute the Taylor series expansion of the function

$$f(z) = \frac{z^2+1}{z+1}$$

near the point $z_0 = 1$ and determine the radius of convergence of the series.

Problem 4. (a) (5 pts) The function $f = f(x)$ is defined by $f(x) = |x|$ for $|x| < 1$ and $f(x) = 0$ for $|x| \geq 1$. Denote by $I_f(x)$ the Fourier integral of f. Sketch the graph of I_f and compute $I_f(1)$.

(b) (5 pts) Let $f(x) = 2x$, $|x| < 1$. Denote by $S_f(x)$ the sum of the Fourier series of the periodic extension of f with period 2. Sketch the graph of S_f and compute $S_f(3)$.

(c) (5 pts) The Fourier transform of the function $f(x) = \dfrac{1}{1+x^2}$ is $\widehat{f}(\omega) = \sqrt{\dfrac{\pi}{2}}\, e^{-|\omega|}$. Use the result to compute the Fourier transform of the function $f(x) = \dfrac{2}{5+3x^2}$

(d) (5 pts) The Fourier transform of $f(x) = \dfrac{1}{1+x^2}$ is $\widehat{f}(\omega) = \sqrt{\dfrac{\pi}{2}}\, e^{-|\omega|}$. Use the result to evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$.

Problem 5. (15 points) Let f be a 2π periodic function so that $f(x) = 1, 0 \le x < \pi; f(x) = 0, \pi \le x < 2\pi$. The Fourier series of f is

$$S_f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k \ge 0} \frac{\sin((2k+1)x)}{2k+1}.$$

(a) Use the result to evaluate the sum $\displaystyle\sum_{k \ge 0} \frac{(-1)^k}{2k+1}$.

(b) Use the result to evaluate the sum $\displaystyle\sum_{k \ge 0} \frac{1}{(2k+1)^2}$.

(c) Use the result to compute the Fourier series of the function $g(x) = 1, 0 \le x \le 1/4$; $g(x) = -1, 1/4 < x < 1/2$, g is periodic with period 1.

Problem 6. (10 points) Use separation of variables to find a non-constant solution $u = u(t, x)$ of the partial differential equation $u_t + 2uu_x = 0$.

Problem 7. (15 points) Solve the following initial-boundary value problem:

$$\begin{aligned} u_t &= u_{xx},\ u = u(x,t),\ x \in (0,1),\ t > 0, \\ u(x,0) &= \sum_{n \ge 0} \frac{\sin(2\pi(2n+1)x)}{(2n+1)^2}, \\ u(0,t) &= 0, \\ u(1,t) &= 0. \end{aligned}$$

Problem 8. (10 points) Solve the following initial value problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0,\ u = u(x,t),\ x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \sin x, \\ u_t(x,0) &= 0. \end{aligned}$$

and determine the set of points (x, t) such that $u(x, t) = 0$.

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FROM 2020

Note: Table summarizing properties of the Fourier transform was included.

Problem 1. Compute the flux of the vector field $\mathbf{F} = (3x - 4xz)\,\hat{\boldsymbol{\imath}} + (z^3 + 2y)\,\hat{\boldsymbol{\jmath}} + (4z + xy)\,\hat{\boldsymbol{\kappa}}$ out of the sphere $x^2 + y^2 + z^2 = 1$.

Problem 2. Compute a polynomial solution of the equation

$$zw''(z) + (1-z)w'(z) + 2w(z) = 0.$$

$(2-4x+x^2)/2$: `Laguerre`

Problem 3. Compute the residue of the function $f(z)=\dfrac{2z+1}{(z+3)^2}$ at the point $z_0=-3$.

Problem 4. Compute the Taylor series expansion of the function

$$f(z)=\frac{z^2+1}{z+1}$$

near the point $z_0=1$ and determine the radius of convergence of the series.

Problem 5. The function $f=f(x)$ is defined by $f(x)=\cos x$ for $|x|<\pi$ and $f(x)=0$ for $|x|\geq\pi$. Denote by $I_f(x)$ the Fourier integral of f. Sketch the graph of I_f for $x\in[-2\pi,2\pi]$ and compute $I_f(\pi)$.

Problem 6. Let $f(x)=2x$, $|x|<2$. Denote by $S_f(x)$ the sum of the Fourier series of the periodic extension of f with period 4. Sketch the graph of S_f for $x\in[-6,6]$ and compute $S_f(3)$.

Problem 7. The Fourier transform of the function $f(x)=e^{-x^2/2}$ is $\widehat{f}(\omega)=e^{-\omega^2/2}$. Compute the Fourier transform of the function

$$g(x)=(2x^2-1)e^{-x^2/2}.$$

`The answer is` $-g(\omega)$

Problem 8. Use separation of variables to find a non-constant solution $u=u(t,x)$ of the equation $uu_t=(u_x)^2$.

`remarkably, both` e^{t+x} `and` e^{t-x} `work`

Problem 9. Solve the initial value problem

$$\begin{aligned} u_{tt} &= 9u_{xx},\ u=u(x,t),\ x\in\mathbb{R},\ t>0,\\ u(x,0) &= \sin x,\\ u_t(x,0) &= 0. \end{aligned}$$

Problem 10. Solve the initial-boundary value problem

$$\begin{aligned} u_t &= \tfrac{1}{2}u_{xx},\ u=u(t,x),\ t>0,\ x\in(0,1),\\ u(0,x) &= \sin(\pi x)-3\sin(4\pi x),\\ u(t,0) &= 0,\\ u(t,1) &= 0. \end{aligned}$$