MATH 445: Preparation for Mid-Term Exam 2
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The tables on page 3 will be available during the exam; $\mathfrak{i}=\sqrt{-1}$.

Problem 1: Computing Fourier transform.

Example. The Fourier transform of $f(x)=e^{-|x|}$ is $\hat{f}(\omega)=\sqrt{2/\pi}(1+\omega^2)^{-1}$. Use the result to compute the Fourier transform of $g(x)=xe^{-|x|/2}$.

Reasoning. If $h(x)=e^{-|x|/2}$, then $g(x)=xh(x)$ and $h(x)=f(x/2)$, so that $\hat{h}(\omega)=2\hat{f}(2\omega)$ and then $\hat{g}(\omega)=\mathfrak{i}\hat{h}'(\omega)$.

Computations. $\hat{h}(\omega)=2^{3/2}\pi^{-1/2}(1+4\omega^2)^{-1}$, $\hat{g}(\omega)=-8\mathfrak{i}\omega 2^{3/2}\pi^{-1/2}(1+4\omega^2)^{-2}$.

Now, do the following. The Fourier transform of the function $f(x)=e^{-x^2/2}$ is $\widehat{f}(\omega)=e^{-\omega^2/2}$. Compute the Fourier transform of the function $g(x)=xe^{-x^2}$. [the answer is $-\mathfrak{i}\omega 2^{-3/2}e^{-\omega^2/4}$]

Problem 2: Plancherel's identity.

Example. The Fourier transform of $f(x)=e^{-|x|}$ is $\hat{f}(\omega)=\sqrt{2/\pi}(1+\omega^2)^{-1}$. Us the result to compute $I=\int_0^{+\infty}(1+\omega^2)^{-2}\,d\omega$.

Reasoning. By Plancherel, $\int_{-\infty}^{+\infty}|f(x)|^2\,dx=\int_{-\infty}^{+\infty}|\hat{f}(\omega)|^2\,d\omega$.

Computation. $2I=(\pi/2)\int_{-\infty}^{+\infty}|\hat{f}(\omega)|^2\,d\omega=(\pi/2)\int_{-\infty}^{+\infty}e^{-2|x|}\,dx=\pi\int_0^{+\infty}e^{-2x}dx=\pi/2$ so $I=\pi/4$.

Now do the following. The Fourier transform of the function

$$f(t)=\begin{cases}1, & \text{if } |t|\le 1,\\ 0, & \text{if } |t|>1\end{cases}$$

is $\widehat{f}(\omega)=\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$. Use the result to evaluate the integral $I=\displaystyle\int_0^\infty\left(\frac{\sin\omega}{\omega}\right)^2d\omega$. [the answer is $\pi/2$.]

Problem 3: Basic separation of variables.

Example. Use separation of variables to find a non-constant solution $u=u(t,x)$ of the equation $u_t=u^6u_x$ such that the function $u=u(t,x)$ is defined for all $x>-1$ and $t<1$.

Reasoning. Write $u(t,x)=f(t)g(x)$ and plug into the equation: $f'g=f^6g^6fg'$ or $(f'/f^7)(t)=(g^5g')(x)$. For equality to hold for all t and x, both expressions must be the same number a; make it $a=1$. This leads to two ODEs.

Computations. $f'/f^7=1$ means $-f^{-6}(t)=(t+c_1)/6$, and, because even powers are non-negative, we must have $t+c_1\le 0$. Thus, for f to be defined for $t<1$, we need $c_1\le -1$, so take $c_1=-1$ and then $f(t)=((1-t)/6)^{-1/6}$. Similarly, $g^5g'=1$ means $g^6(x)=(x+c_2)/6\ge 0$; for g to be defined for $x>-1$ we need $c_2\ge 1$, so take $c_2=1$ and then $g(x)=((x+1)/6)^{1/6}$. Because $u=fg$, we get the final answer $u(t,x)=\left(\frac{x+1}{1-t}\right)^{1/6}$.

Now solve the same problem for $u_t=u^4u_x$. [a possible answer is $\left(\frac{x+1}{1-t}\right)^{1/4}$]

Problem 4: Transport equation.

Example. Solve the following initial value problem:

$$\begin{aligned} u_t-5u_x &= 0,\ u=u(t,x),\ t>0,\ x\in\mathbb{R},\\ u(0,x) &= e^x. \end{aligned}$$

Reasoning. Here, we just know that the solution of $u_t+cu_x=0$ is $u(0,x-ct)$.

Solution. With $c=-5$ and $u(0,x)=e^x$, the answer is $u(t,x)=e^{x+5t}$.

Now solve the following initial value problem:

$$\begin{aligned} u_t + 5u_x &= 0, \ u = u(t,x), \ t > 0, \ x \in \mathbb{R}, \\ u(0,x) &= \cos x. \end{aligned}$$

[the answer is $\cos(x - 5t)$.]

Problem 5: Wave equation on an interval

Example. Solve the following initial-boundary value problem:

$$\begin{aligned} u_{tt} &= 9u_{xx}, \ u = u(t,x), \ t > 0, \ x \in (0,\pi), \\ u(0,x) &= \sin(3x) - 3\sin(4x), \\ u_t(0,x) &= 0, \\ u(t,0) &= 0, \\ u(t,\pi) &= 0. \end{aligned}$$

Reasoning. From the boundary conditions, we expect the solution to be $\sum_{k\geq 1} f_k(t)\sin(kx)$ and, by linearity, each $f_k(t)\sin(kx)$ should satisfy the equation, that is, $f_k''(t) = -9k^2 f_k(t)$. The solution of this equation is $f_k(t) = f_k(0)\cos(3kt) + (f_k'(0)/(3k))\sin(3kt)$. From initial conditions, $f_3(0) = 1, \ f_4(0) = -3$, and $f_k(0) = 0$ for all other k; $f_k'(0) = 0$ for all k.

Computation. We write down the final answer: $u(t,x) = \cos(9t) \ \sin(3x) - 3\cos(12t) \ \sin(4x)$.

Now solve the following initial-boundary value problem:

$$\begin{aligned} u_{tt} &= 9u_{xx}, \ u = u(t,x), \ t > 0, \ x \in (0,\pi), \\ u(0,x) &= 0, \\ u_t(0,x) &= \sin(3x) - 3\sin(4x), \\ u(t,0) &= 0, \\ u(t,\pi) &= 0. \end{aligned}$$

[the answer is $(1/9)\sin(9t) \ \sin(3x) - (1/4)\sin(12t) \ \sin(4x)$].

Properties of the Fourier series and transform

Series	Name	Transform
$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}\,dx$	Forward	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-\mathrm{i}x\omega}\,dx$
$S_f(x) = \sum\limits_{k=-\infty}^{+\infty} c_k(f)e^{\mathrm{i}kx}$	Inverse	$I_f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} \hat{f}(\omega)e^{\mathrm{i}x\omega}\,d\omega$
$c_0(f) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)\,dx$	Obvious	$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,dx$
$\sum\limits_{k=-\infty}^{+\infty} c_k(f) = S_f(0) = \frac{\tilde{f}(0+)+\tilde{f}(0-)}{2}$	Obvious	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\omega)\,d\omega = I_f(0) = \frac{f(0+)+f(0-)}{2}$
$\lim\limits_{\lvert k\rvert\to\infty} \lvert c_k(f)\rvert = 0$	Riemann-Lebesgue: $f \in L_1$	$\lim\limits_{\lvert\omega\rvert\to\infty} \lvert\hat{f}(\omega)\rvert = 0,\ \hat{f}$ continuous
$\sum\limits_{k=-\infty}^{+\infty} \lvert c_k(f)\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert f(x)\rvert^2\,dx$	Parseval/Plancherel: $f \in L_2$	$\int\limits_{-\infty}^{+\infty} \lvert\hat{f}(\omega)\rvert^2\,d\omega = \int\limits_{-\infty}^{+\infty} \lvert f(x)\rvert^2\,dx$

Further properties of the Fourier transform

Function	Fourier transform	Function	Fourier transform
$f(x)$	$\hat{f}(\omega) = \mathcal{F}[f](\omega)$	$\hat{f}(x)$	$f(-\omega)$
$f(x-a)$	$e^{-\mathrm{i}a\omega}\hat{f}(\omega)$	$e^{\mathrm{i}ax}f(x)$	$\hat{f}(\omega-a)$
$f(x/\sigma)$	$\sigma\hat{f}(\sigma\omega)$	$e^{-x^2/2}$	$e^{-\omega^2/2}$
$f'(x)$	$\mathrm{i}\omega\hat{f}(\omega)$	$xf(x)$	$\mathrm{i}\hat{f}'(\omega)$
$f''(x)$	$-\omega^2\hat{f}(\omega)$	$x^2f(x)$	$-\hat{f}''(x)$
$\int f(x)\,dx$	$\frac{\hat{f}(\omega)}{\mathrm{i}\omega}$	$\frac{f(x)}{x}$	$\frac{1}{\mathrm{i}}\int \hat{f}(\omega)\,d\omega$
$(f*g)(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$	$f(x)g(x)$	$\frac{1}{\sqrt{2\pi}}(\hat{f}*\hat{g})(\omega)$
$e^{-\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$	$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}}e^{-\lvert\omega\rvert}$
$1(\lvert x\rvert \le 1)$	$\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$	$\frac{\sin\omega}{\omega}$	$\sqrt{\frac{\pi}{2}}1(\lvert x\rvert \le 1)$
$\delta_a(x)$	$e^{-\mathrm{i}\omega a}/\sqrt{2\pi}$	$\cos(ax)$	$\sqrt{\pi/2}\Big(\delta_a(\omega)+\delta_{-a}(\omega)\Big)$