MATH 445 Final Exam, Fall 2022

Instructor — S. Lototsky (KAP 248D; x0–2389; lototsky@usc.edu)

Instructions:

- No notes, no books or other printed materials (including printouts from the web), no calculators, no collaboration with anybody.
- Answer all questions, show your work (when appropriate), upload solutions to Gradescope.
- **There are nine problems. Each of the problems 1–8 is worth 10 points. Problem 9 is a collection of 10 multiple choice questions**
- $e^z = 1 + z + z^2/(2!) + z^3/(3!) + \cdots$; $\sin(z) = z - z^3/(3!) + z^5/(5!) - \cdots$; $\cos(z) = 1 - (z^2/2!) + (z^4/4!) + \cdots$.

Problem 1.

(a) [5 pts] Compute the line integral $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y,z) = 2x^3y^5z^7$, ∇f is the gradient of f, and C is a straight line segment from the point $(0,0,0)$ to the point $(1,1,1)$. $f(1,1,1) - f(0,0,0) = 2$

(b) [5 pts] Compute the flux of the vector field $\mathbf{F} = (3x + 2xy)\,\hat{\imath} + (z^2 - y^2)\,\hat{\jmath} + (4 + x)z\,\hat{\kappa}$ out of the sphere $(x+1)^2 + (y-3)^2 + (z-1)^2 = 1$. $\operatorname{div}\vec{F} = 3 + 2y - 2y + 4 + x = 7 + x \Rightarrow \frac{4\pi}{3}(7 + \underbrace{\bar{x}}_{=-1}) = \frac{4\pi}{3}\cdot 6 = 8\pi$

Problem 2.

(a) [5 pts] Compute $\oint_C \frac{\boxed{e^{2z} - 1}}{z^2}dz$, where C is the circle $|z| = 5$, oriented counterclockwise. $e^{2z}-1 \approx 2z \Rightarrow 2\cdot 2\pi i = 4\pi i$

(b) [5 pts] Compute the Laurent series expansion of the function $f(z) = \frac{z+5}{z-5}$ around the point $z_0 = 5$. $= 1 + \frac{10}{z-5}$

Problem 3. Solve the initial value problem

Hermite

$\Rightarrow y = bx^3 + ax$, $y' = 3bx^2 + a$, $y'' = 6bx$, $a = -3b$

$$y''(x) - xy'(x) + 3y(x) = 0, \quad y(0) = 0, \quad y'(0) = -3,$$

$6bx - 3bx^3 - ax + 3bx^3 + 3ax$, $a = -3$

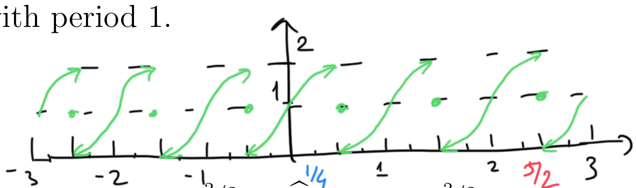
$\boxed{y(x) = x^3 - 3x}$.

Problem 4. Let $f(x) = 1 + \sin(\pi x)$, $|x| < 1/2$, and let $S_f = S_f(x)$, $x \in (-\infty, +\infty)$ be the sum of the Fourier series of the periodic extension of f with period 1.

(a) [4pt] Draw the graph of S_f for $x \in [-3, 3]$;

(b) [3pt] Compute $S_f(5/4) = f(1/4) = 1 + \sqrt{2}/2$

(c) [3pt] Compute $S_f(5/2) = \frac{1}{2}(0+2) = 1.$



Problem 5. The Fourier transform of the function $f(x) = e^{-x^2/2}$ is $\hat{f}(\omega) = e^{-\omega^2/2}$. Compute the Fourier transform of the function $g(x) = e^{-(x-1)^2}$. $\hat{g}(\omega) = \frac{1}{\sqrt{2}}\exp(-i\omega - \frac{\omega^2}{4})$

$g(x) = h(x-1)$; $h(x) = f(\sqrt{2}\,x)$

Problem 6. Use separation of variables to find a non-constant solution $u = u(t,x)$ of the equation $u_t = u^2u_x$ such that the function $u = u(t,x)$ is defined for all $x > -2$ and $t < 2$. $u = f(t)g(x)$

$\boxed{u = \left(\frac{a+x}{a-t}\right)^{1/2},\ a \geq 2.}$

$f'g = f^2g^2fg'$

$\frac{f'}{f^3} = gg' = 1$

Problem 7. Solve the following initial value problem:

$$\begin{aligned} u_t - 5u_x &= 0, \ u = u(t,x), \ t > 0, \ x \in \mathbb{R}, \\ u(0,x) &= \cos x. \end{aligned}$$

Transport: $u(t,x) = \cos(x + 5t)$

Problem 8. Solve the following initial-boundary value problem:

$$\begin{aligned} u_{tt} &= \overset{=2^2}{4}u_{xx}, \ u = u(t,x), \ t > 0, \ x \in (0,\pi), \\ u(0,x) &= 0, \\ u_t(0,x) &= \sin(2x) + 3\sin(5x), \\ u(t,0) &= 0, \\ u(t,\pi) &= 0. \end{aligned}$$

(*) $C = 2 \Rightarrow$ by (*), $\sin(kx)\sin(2kt)$

$\Downarrow$

$$u(t,x) = \frac{\sin(2x)\sin(4t)}{4} + \frac{3\sin(5x)\sin(10t)}{10}$$

Problem 9. This is a multiple choice part. For each of the 10 questions, circle (or otherwise indicate) the answer you think is correct (there is always only one correct answer). You get two points for each correct selection, zero points for each wrong selection. No need to show your work.

(i) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors. Which ONE of the following expressions is always equal to zero?

$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{b}$ $\boldsymbol{a} \cdot \boldsymbol{a}$ $(\boldsymbol{b} \times \boldsymbol{a}) \times \boldsymbol{b}$ $\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})$

(ii) Let f be a scalar field and $\boldsymbol{F}$, a vector field. Assuming that all the necessary partial derivatives exist and are continuous, circle the ONE expression that is always equal to zero.

$\boldsymbol{F} \cdot \underbrace{\mathrm{curl}\big(\mathrm{grad}(f)\big)}_{=0}$ $\mathrm{grad}\big(\mathrm{div}(f\boldsymbol{F})\big)$ $\mathrm{curl}\big(\mathrm{curl}(f\boldsymbol{F})\big)$ $\mathrm{grad}\big((\mathrm{grad}\, f) \cdot \boldsymbol{F}\big)$

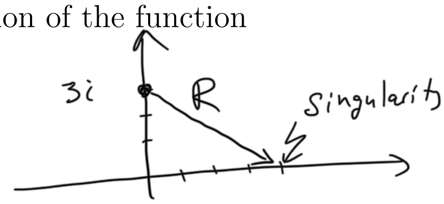
$\sim -\frac{z^2}{2}$

(iii) What is the type of singularity of the function $f(z) = \dfrac{\cos(z) - 1}{z^2}$ at the point $z = 0$?

Removable   Simple pole   Pole of order 2   Essential

(iv) What is the radius of convergence of the Taylor series expansion of the function $f(z) = \dfrac{z + 5\sin z}{z - 4}$ around the point $z_0 = 3i$? $R = \sqrt{3^2 + 4^2}$

2   3   4   5   6



(v) Circle the function $u = u(x, y)$ that is harmonic

$u = x^2 - 2y^2$ $u = e^x \sin(y)$ $u = x^2 + y^2$ $u = xe^y$

(vi) True or false: the Fourier series of the 2π periodic extension of the function $f(x) = x$, $|x| < \pi$, converges uniformly on the interval $[-10, 10]$?

True   False   Need more information

(vii) Identify the wave equation

transport	Laplace	Heat	wave	Burgers
$u_t = u_x$	$u_{xx} + u_{yy} = 0$	$u_t = u_{xx}$	$u_{tt} = u_{xx}$	$u_t + uu_x = 0$

(viii) Identify the sequence that converges uniformly to zero on the interval $x \in [0, 1]$, as $n \to \infty$:

x[illegible] $n\sin($[illegible]$)$ $nx/($[illegible]$\, nx)$ $x^2/(n + x) \leq \frac{1}{n}$.

(ix) Identify the function $f = f(x)$ for which the Fourier transform $\hat{f}$ is real [that is, the imaginary part of $\hat{f}$ is identically equal to zero.] Need $f(x) = f(-x)$

$e^{-x^2 + x\sin(2x)}$ $e^{-|x|}\sin(2x)$ $e^{-x^4 + x^3}$ $e^{-x^2}\big(\cos(2x) + \sin(x)\big)$

(x) Let f be a scalar field for which all partial derivatives exist and are continuous. Which of the following expressions defines the Laplacian $\boldsymbol{\Delta} f$ of f?

$\mathrm{curl}\big(\mathrm{grad}(f)\big)$ $\mathrm{div}\big(\mathrm{grad}(f)\big) = f_{xx} + f_{yy} + f_{zz}$ $\mathrm{grad}\big(\mathrm{div}(f)\big)$ $\mathrm{grad}\big(|\,\mathrm{grad}\, f|^2\big)$