# Spring 2013, MATH 245, Final Exam

## Friday, May 10, 2013;  8–10am

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**Name:** ______________________________

**Instructions:**

- No notes, books, calculators, etc.
- Answer all questions and clearly indicate your answers.
- **Show your work!** Points might be taken off for correct answer with no explanations. Wrong answer with no explanations is worth zero points.

| Problem | Possible | Actual | Problem | Possible | Actual |
|---|---|---|---|---|---|
| 1 | 10 | | 6 | 10 | |
| 2 | 10 | | 7 | 10 | |
| 3 | 10 | | 8 | 10 | |
| 4 | 10 | | 9 | 10 | |
| 5 | 10 | | 10 | 10 | |
| Total | 50 | | Total | 50 | |

Properties of the Laplace transform

| Function | Laplace transform | Function | Laplace transform |
|---|---|---|---|
| $f(t)$ | $F(s) = \int_0^{+\infty} e^{-st} f(t) dt$ | $1 = u_0(t)$ | $\frac{1}{s}$ |
| $af(t) + bg(t)$ | $aF(s) + bG(s)$ | $\delta(t)$ | $1$ |
| $e^{-ct} f(t)$ | $F(s+c)$ | $e^{at}$ | $\frac{1}{s-a}$ |
| $f(t-c) = f(t-c)u_c(t)$ | $e^{-cs}F(s),\ (c > 0)$ | $\sin(at)$ | $\frac{a}{s^2+a^2}$ |
| $f'(t)$ | $sF(s) - f(0)$ | $\cos(at)$ | $\frac{s}{s^2+a^2}$ |
| $f''(t)$ | $s^2F(s) - sf(0) - f'(0)$ | $\frac{t}{2a}\sin(at)$ | $\frac{s}{(s^2+a^2)^2}$ |
| $\int_0^t f(\tau) d\tau$ | $\frac{F(s)}{s}$ | $\frac{1}{2a^3}\big(\sin(at) - at\cos(at)\big)$ | $\frac{1}{(s^2+a^2)^2}$ |
| $tf(t)$ | $-F'(s)$ | $t^r,\ r > -1$ | $\frac{\Gamma(r+1)}{s^{r+1}}$ |
| $t^n f(t)$ | $(-1)^n F^{(n)}(s)$ | $t^n,\ n = 1, 2, 3, \ldots$ | $\frac{n!}{s^{n+1}}$ |
| $\frac{f(t)}{t}$ | $\int_s^{+\infty} F(z) dz$ | $f(t+T) = f(t),\ T > 0$ | $\frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$ |
| $f(ct),\ c > 0$ | $\frac{1}{c}F(s/c)$ | $\int_0^t f(t-\tau)g(\tau) d\tau$ | $F(s)G(s)$ |

**Problem 1.** For the equation $y' = y^2(y-1)(y+2)$, sketch the integral curves and classify each equilibrium solution as asymptotically stable, unstable, or neither.

**Problem 2**. Solve the initial value problem $y' = \dfrac{xy^3}{\sqrt{1+x^2}}$, $y(0) = 1$. Write your answer in the form $y = y(x)$.

**Problem 3**. Solve the initial value problem $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$, $y(1) = 1$. Write your answer in the form $y = y(x)$.

**Problem 4**. Solve the initial value problem $y'' - 2y' - 3y = 3e^{2t}$, $y(0) = 1, y'(0) = -1$, using (a) undetermined coefficients (b) Laplace transform. Make sure you get the same answers.

**Problem 5**. Compute the Laplace transform of $t^3 e^{-t} \sin t$.
Suggestion: use $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ and take the imaginary part of a suitable expression.

**Problem 6**. Compute the inverse Laplace transform of $\dfrac{5s}{s^2 - s - 6}$.

**Problem 7.** Find the general solution of the system

$$\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

**Problem 8**. Determine all values of the parameter $c$ for which all solutions of the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & c \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

are asymptotically stable.

**Problem 9.** Sketch the phase portrait for the equation $y'' + 2y' + y = 0$.

**Problem 10**. For the system,

$$\begin{cases} x' = -y + xy \\ y' = 3x - x^2 - xy \end{cases}$$

determine the location and type of all critical points.