Mathematics of Physics and Engineering I (Math 245): Computer project

The project is due `as a hard copy` at the last lecture of the semester. During the first week of classes, you will learn most of what you need to do the project.

Your task is to investigate the following numerical approximation of the solution of the equation
$y'(x) = ay,\ y(0) = 1,\ x \in [0, 1]$:

$$y_{n+1} = y_n + a\triangle x(\lambda y_n + (1-\lambda)y_{n+1}),\ n = 0, \ldots, N-1,\ y_0 = 1,$$

where the real number a is a parameter in the equation, real number λ is a parameter of the numerical approximation, and $\triangle x = 1/N$ is the (uniform) step size.

Two questions to address: (a) is it easier to approximate the solution when $a < 0$ or when $a > 0$? (b) Is there a value of λ that you would consider the best choice, and how does this best value depend on a and on the way you measure the approximation error?

Note: (a) It is known that $\lambda = 1/2$ is supposed to be the best in some sense, but your task is to investigate whether it is indeed always the best; (b) there are different ways to quantify how easy it is to compute the approximation, but it is important to keep the comparison "fair", so use your imagination and common sense; (c) there are two distinct ways to measure the approximation error, using either absolute error, measuring the difference between the exact and approximate values, or using the relative error, which is the ratio of the absolute error to the absolute value of the exact solution. The standard measures of the absolute error are

$$\varepsilon_p = \begin{cases} \left(\frac{1}{N}\sum_{k=1}^{N} |y_k - y(k/N)|^p\right)^{1/p}, & p \leq 1 < \infty \\ \max_{k=1,\ldots,N} |y_k - y(k/N)|, & p = \infty. \end{cases}$$

Given the variety of ways one can measure the error, it is not clear a priori that the same value of λ will always be the best.

You are welcome to pose and investigate other questions as well. For example, nobody said that λ must be between 0 and 1, and the conclusions might also depend on the initial condition.

Please support your conclusions with (clearly readable) graphs and/or tables; theoretical analysis is also welcome, and (reasonable) experimental deviations from theoretical predictions are especially encouraged.

It is my intention not to give detailed guidelines for the project: this is your chance to be creative. There is plenty for you to investigate. Your questions and comments are always welcome.

Please note:

- You are welcome to use any software package and any help, including any on-line resource you can find.
- If you are not comfortable with computers, start early and seek help often.