4. Let $X = (X_n, \mathscr{F}_n)_{n\geq 0}$ be a submartingale such that $X_n \geq \mathsf{E}(\xi|\mathscr{F}_n)$ (P-a.s.), $n \geq 0$, where $\mathsf{E}|\xi| < \infty$. Show that if τ_1 and τ_2 are stopping times with $\mathsf{P}(\tau_1 \leq \tau_2) = 1$, then

$$X_{\tau_1} \geq \mathsf{E}(X_{\tau_2}|\mathscr{F}_{\tau_1}) \quad (\text{P-a.s.}).$$

5. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables with $\mathsf{P}(\xi_i = 1) = \mathsf{P}(\xi_i = -1) = \frac{1}{2}$, a and b positive numbers, $b > a$,

$$X_n = a \sum_{k=1}^{n} I(\xi_k = +1) - b \sum_{k=1}^{n} I(\xi_k = -1)$$

and

$$\tau = \inf\{n \geq 1: X_n \leq -r\}, \qquad r > 0.$$

Show that $\mathsf{E}e^{\lambda\tau} < \infty$ for $\lambda \leq \alpha_0$ and $\mathsf{E}e^{\lambda\tau} = \infty$ for $\lambda > \alpha_0$, where

$$\alpha_0 = \frac{b}{a+b}\ln\frac{2b}{a+b} + \frac{a}{a+b}\ln\frac{2a}{a+b}.$$

6. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables with $\mathsf{E}\xi_i = 0$, $\mathsf{V}\xi_i = \sigma_i^2$, $S_n = \xi_1 + \cdots + \xi_n$, $\mathscr{F}_n^\xi = \sigma\{\omega: \xi_1, \ldots, \xi_n\}$. Prove the following generalizations of Wald's identities (14) and (15): If $\mathsf{E}\sum_{j=1}^{\tau} \mathsf{E}|\xi_j| < \infty$ then $\mathsf{E}S_\tau = 0$; if $\mathsf{E}\sum_{j=1}^{\tau} \mathsf{E}\xi_j^2 < \infty$, then

$$\mathsf{E}S_\tau^2 = \mathsf{E}\sum_{j=1}^{\tau} \xi_j^2 = \mathsf{E}\sum_{j=1}^{\tau} \sigma_j^2. \tag{23}$$

§3. Fundamental Inequalities

1. Let $X = (X_n, \mathscr{F}_n)_{n\geq 0}$ be a stochastic sequence,

$$X_n^* = \max_{0\leq j\leq n} |X_j|, \; \|X_n\|_p = (\mathsf{E}|X_n|^p)^{1/p}, \qquad p > 0.$$

In Theorems 1–3 below, we present Doob's fundamental "maximal inequalities for probabilities" and "maximal inequalities in L^p," for submartingales, supermartingales and martingales.

Theorem 1. I. *Let $X = (X_n, \mathscr{F}_n)_{n\geq 0}$ be a submartingale. Then for all $\lambda > 0$*

$$\lambda \mathsf{P}\left\{\max_{k\leq n} X_k \geq \lambda\right\} \leq \mathsf{E}\left[X_n^+ I\left(\max_{k\leq n} X_k \geq \lambda\right)\right] \leq \mathsf{E}X_n^+, \tag{1}$$

$$\lambda \mathsf{P}\left\{\min_{k\leq n} X_k \leq -\lambda\right\} \leq \mathsf{E}\left[X_n I\left(\min_{k\leq n} X_k > -\lambda\right)\right] - \mathsf{E}X_0 \leq \mathsf{E}X_n^+ - \mathsf{E}X_0, \tag{2}$$

$$\lambda \mathsf{P}\left\{\max_{k\leq n} |X_k| \geq \lambda\right\} \leq 3 \max_{k\leq n} \mathsf{E}|X_k|. \tag{3}$$

II. *Let $Y = (Y_n, \mathscr{F}_n)_{n\geq 0}$ be a supermartingale. Then for all $\lambda > 0$*

$$\lambda \mathsf{P}\left\{\max_{k\leq n} Y_k \geq \lambda\right\} \leq \mathsf{E}Y_0 - \mathsf{E}\left[Y_n I\left(\max_{k\leq n} Y_k < \lambda\right)\right] \leq \mathsf{E}Y_0 + \mathsf{E}Y_n^-, \tag{4}$$

$$\lambda \mathsf{P}\left\{\min_{k\leq n} Y_k \leq -\lambda\right\} \leq -\mathsf{E}\left[Y_n I\left(\min_{k\leq n} Y_k \leq -\lambda\right)\right] \leq \mathsf{E}Y_n^-, \tag{5}$$

$$\lambda \mathsf{P}\left\{\max_{k\leq n} |Y_k| \geq \lambda\right\} \leq 3 \max_{k\leq n} \mathsf{E}|Y_k|. \tag{6}$$

III. *Let $Y = (Y_n, \mathscr{F}_n)_{n\geq 0}$ be a nonnegative supermartingale. Then for all $\lambda > 0$*

$$\lambda \mathsf{P}\left\{\max_{k\leq n} Y_k \geq \lambda\right\} \leq \mathsf{E}Y_0, \tag{7}$$

$$\lambda \mathsf{P}\left\{\sup_{k\geq n} Y_k \geq \lambda\right\} \leq \mathsf{E}Y_n. \tag{8}$$

Theorem 2. *Let $X = (X_n, \mathscr{F}_n)_{n\geq 0}$ be a nonnegative submartingale. Then for $p \geq 1$ we have the following inequalities:*

if $p > 1$,

$$\|X_n\|_p \leq \|X_n^*\|_p \leq \frac{p}{p-1}\|X_n\|_p; \tag{9}$$

if $p = 1$,

$$\|X_n\| \leq \|X_n^*\|_1 \leq \frac{e}{e-1}\{1 + \|X_n \ln^+ X_n\|_1\}. \tag{10}$$

Theorem 3. *Let $X = (X_n, \mathscr{F}_n)_{n\geq 0}$ be a martingale, $\lambda > 0$ and $p \geq 1$. Then*

$$\mathsf{P}\left\{\max_{k\leq n} |X_k| \geq \lambda\right\} \leq \frac{\mathsf{E}|X_n|^p}{\lambda^p} \tag{11}$$

and if $p > 1$

$$\|X_n\|_p \leq \|X_n^*\|_p \leq \frac{p}{p-1}\|X_n\|_p. \tag{12}$$

In particular, if $p = 2$

$$\mathsf{P}\left\{\max_{k\leq n} |X_k| \geq \lambda\right\} \leq \frac{\mathsf{E}|X_n|^2}{\lambda^2}, \tag{13}$$

$$\mathsf{E}\left[\max_{k\leq n} X_k^2\right] \leq 4\mathsf{E}X_n^2. \tag{14}$$