Spring 2023, MATH 408, Exam 1

Monday, March 6; 11–11:50am

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Instructions:

- You should have access to a calculator or some other computing device, and to the normal and χ^2 distribution tables. Instead of the tables, you are welcome to use the statistical functions available on your computing device.
- Answer all questions and clearly indicate your answers; upload the solutions to GradeScope.
- **Each problem is worth 10 points.**

Problem 1. Let $X_1, \dots, X_{25}$ be an independent random sample from a normal population with unknown mean μ and unknown variance σ^2. It is known that

$$\sum_{k=1}^{25} X_k = 250, \quad \sum_{k=1}^{25} X_k^2 = 3100.$$

$\overline{X}_n = \frac{250}{25} = 10;\ S_n = 5$

$S_n^2 = \frac{1}{24}(3100 - 25\cdot 10^2) = 25$

χ^2_{24} 0.025 0.025 12.401 39.364 $(n-1)S_n^2 = 600$

$\left[\sqrt{\frac{600}{39.364}}, \sqrt{\frac{600}{12.401}}\right] \approx \boxed{[4, 7]}$

Construct the 95% confidence interval for the standard deviation σ.

To get full credit, indicate the values of the sample mean, sample standard deviation, and the quantile of the corresponding distribution you need to construct the confidence interval.

Problem 2. Let $X_1, \dots, X_n$ be an independent random sample from the $Gamma(5, \theta)$ distribution; in particular, the pdf of each X_k is

$$f(x;\theta) = \frac{1}{24\theta^5} x^4 e^{-x/\theta} 1(x > 0).$$

$L_n = \left(\prod \frac{X_k^4}{24}\right) \exp\left(-\frac{n\overline{X}_n}{\theta} - 5n\ln\theta\right)$

Construct the MLE of θ. Do not forget to check that the critical point is indeed the point of maximum.

$\frac{\partial \ln L_n}{\partial \theta} = \frac{n\overline{X}_n}{\theta^2} - \frac{5n}{\theta}$ + − $\overline{X}_n/5$ $\Rightarrow \boxed{\hat{\theta}_n^{MLE} = \frac{\overline{X}_n}{5}}$

Problem 3. A study reports that freshmen at public universities work 11.2 hours a week for pay, on average, and the s_n is 8.5 hours; at private universities, the average is 9.3 hours and the s_n is 7.2 hours. Assume these data are based on two independent simple random samples, each of size 500. Is the difference between the averages due to chance? Explain your conclusion by stating the corresponding null and alternative hypotheses and computing the p-value. $\boxed{\text{Difference is NOT due to chance.}}$

p-value $= \mathbb{P}\left(|\mathcal{N}(0,1)| > \frac{11.2 - 9.3}{\sigma}\right)$, $\sigma = \sqrt{\frac{(8.5)^2}{500} + \frac{(7.2)^2}{500}}$ $\Rightarrow$ p-value $= \mathbb{P}(|\mathcal{N}(0,1)| > 3.8) < 0.01$

Problems 4. Let $X_1, \dots, X_n$ be an independent random sample from the $Gamma(4, \theta)$ distribution; in particular, the pdf of each X_k is

$$f(x;\theta) = \frac{1}{6\theta^4} x^3 e^{-x/\theta} 1(x > 0).$$

$\frac{L_n(1)}{L_n(2)} = 2^{4n} \exp\left(-\frac{n\overline{X}_n}{2}\right)$

Construct the most powerful test with Type-I error equal to 0.05 for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$. reject H_0 when $\frac{L_n(1)}{L_n(2)}$ is small $\Leftrightarrow n\overline{X}_n$ is large; at level 0.05, reject H_0 if

$n\overline{X}_n \sim Gamma(4n, 1)$ $\quad n\overline{X}_n > Gamma(4n,1)\big|_\alpha$

Problem 5. For the first-year students at a certain university, the correlation between SAT scores and the first-year GPA was 0.68. Assume that the distribution of the scores is jointly normal. Predict the percentile rank on the first-year GPA for a student whose percentile rank on the SAT was 75%.

X Y

z-value for X: $\mathbb{P}(\mathcal{N}(0,1) < z_X) = 0.75 \Rightarrow z_X \approx 0.675$.

$\Rightarrow$ z-value for Y $= \rho z_X \approx 0.46 \Rightarrow$ GPA rank is $\mathbb{P}(\mathcal{N}(0,1) < 0.46) \approx \boxed{67\%}$

(or 68%)