A Summary of Local Time for Semimartingales.[1]

## General Case

**The setting:** $X = X(t),\ t \geq 0$ is a real-valued semimartingale with canonical representation $X(t) = X(0) + M(t) + A(t)$, where $M$ is a (local) martingale and $A$ is a process with bounded variation. *We assume that the quadratic variation $\langle X\rangle = \langle X\rangle_t$ of the continuous martingale component of $X$ is non-zero.* There is the underlying stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, and the usual assumption that the trajectories of $X$ are *càdlàg*: for every $t > 0$, $X(t) = X(t+)$ and $X(t\text{-})$ exists; $\triangle X(t) = X(t) - X(t\text{-})$.

**A Motivation:** If $f = f(x)$ is twice continuously differentiable, then the Itô formula holds for $f\big(X(t)\big)$:

$$f\big(X(t)\big) = f\big(X(0)\big) + \int_{0+}^t f'\big(X(s\text{-})\big)\,dX(s) + \frac{1}{2}\int_{0+}^t f''\big(X(s\text{-})\big)\,d\langle X\rangle_s + \sum_{0<s\leq t}\Big(f\big(X(s)\big) - f\big(X(s\text{-})\big) - f'\big(X(s\text{-})\big)\triangle X(s)\Big).$$

What if the function is not twice continuously differentiable, for example, $f(x) = |x|$?

**The general idea:** The local time is whatever makes the Itô formula work for $f(x) = |x-a|$ for every $a \in \mathbb{R}$.

**Three equivalent definitions** of the `local time` $L = L^x(t)$, $x \in \mathbb{R}$, $t \geq 0$, *under the additional assumption that $X$ is continuous*:

$$L^x(t) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t I\big(x \leq X(s) < x+\varepsilon\big)\,d\langle X\rangle_s; \tag{1}$$

$$\int_0^t f\big(X(s\text{-})\big)\,d\langle X\rangle_s = \int_{\mathbb{R}} f(x)L^x(t)\,dx,\ f \text{ bounded measurable: "occupation time" formula} \tag{2}$$

$$L^x(t) = |X(t)-x| - |X(0)-x| - \int_0^t \operatorname{sgn}\big(X(s)-x\big)\,dX(s) \text{ (Tanaka's formula.)} \tag{3}$$

THE CONVENTION FOR THE SIGNUM FUNCTION AT ZERO IS $\operatorname{sgn}(0) = -1$.

**Basic facts:** If $X$ is a continuous semi-martingale, then

(1) The local time exists and has a modification that is continuous in time and *càdlàg* in $x$, with jumps

$$L^x(t) - L^{x-}(t) = 2\int_0^t I\big(X(s) = x\big)\,dA(s),$$

where $A$ is the bounded variation component of $X$ [Kallenberg, Theorem 19.4];

(2) For each $x \in \mathbb{R}$, the function $t \mapsto L^x(t)$ is non-decreasing [by (1)];

(3) The `Itô-Tanaka` formula

$$f\big(X(t)\big) = f\big(X(0)\big) + \int_0^t f'_-\big(X(s)\big)\,dX(s) + \frac{1}{2}\int_{\mathbb{R}} L^x(t)\,f''(dx),$$

holds for each of the following types of functions $f$:

- $f$ is absolutely continuous and $f'$ has bounded variation;
- $f$ is a difference of two convex functions;

$f'_-$ denotes the left-hand derivative of $f$, and, with $f'$ being of bounded variation, $f''(dx)$ denotes the measure corresponding to the derivative of $f'$.

**Beyond continuous semimartingales**.

- For an arbitrary semimartingale $X$, $L^x(t)$ is defines by

$$L^x(t) = \big|X(t)-x\big| - \big|X(0)-x\big| - \int_{0+}^t \operatorname{sgn}\big(X(s\text{-})-x\big)\,dX(s) - \sum_{0<s\leq t}\Big(\big|X(s)-x\big| - \big|X(s\text{-})-x\big| - \operatorname{sgn}\big(X(s\text{-})-x\big)\triangle X(s)\Big),$$

which follows the idea of making the Itô formula work for the sgn function.

- for a continuous non-random function $X = X(t)$, local time can be defined using the *usual* occupation time/occupation measure formula

$$\int_0^t f\big(X(s)\big)\,ds = \int_{\mathbb{R}} f(x)T(t,dx), \tag{4}$$

[1] Sergey Lototsky, USC

by setting $L^x(t) = T(t, dx)/dx$ if the density exists in some sense. With this definition, if $X(t) = a$ (constant), then $L^x(t) = t\delta(x-a)$, and if $X$ is continuously differentiable with $X'(t) > 0$, then, after change of variables in (4),

$$L^x(t) = \frac{I\big(X(0) \le x \le X(t)\big)}{X'\big(X^{-1}(x)\big)}.$$

- For a certain type of Markov processes, local time is defined as a particular *additive functional*; see Section 3.6 in M. B. Markus and J. Rosen, Markov processes, Gaussian processes, and local times [Cambridge Studies in Advanced Mathematics, vol. 100. Cambridge University Press, Cambridge, 2006].

For the standard Brownian motion, all the possible constructions of the local time coincide.

## The Case of the Standard Brownian Motion

Because the standard Brownian motion $W = W(t),\ t \ge 0$, is a continuous square-integrable martingale with $\langle W\rangle_t = t$, the corresponding local time $L = L^x(t),\ t \ge 0, x \in \mathbb{R}$, is jointly continuous in $(t, x)$. Moreover, (2) becomes a true occupation time formula:

$$\int_0^t f\big(W(s)\big)\, ds = \int_{\mathbb{R}} f(x) L^x(t)\, dx,$$

which holds even for random $t$, and (3) implies $L^0(t) = |W(t)| - \tilde{W}(t)$, where, by the Lévy characterization of the Brownian motion, $\tilde{W}(t) = \int_0^t \operatorname{sgn}\big(W(s)\big)\, dW(s)$ is a standard Brownian motion.

The Brownian local time is closely connected with the *Bessel processes*, both usual and squared. A `squared Bessel process of order` $\delta \ge 0$ [or the square of a $\delta$-dimensional Bessel process] starting at $x \ge 0$ is the (strong, non-negative) solution of the equation

$$dX(t) = \delta\, dt + 2\sqrt{X(t)}\, dW(t),\ t > 0, \quad X(0) = x;$$

for $\delta = \mathrm{d} = 1, 2, 3, \ldots$, the process describes the *square* of the Euclidean norm of the d-dimensional Brownian motion. Then the process $t \mapsto \sqrt{X(t)}$ is called the $\delta$`-dimensional Bessel process` starting at $\sqrt{x}$.

Below are some remarkable facts related to the Brownian local time $L$:

1 [P. Lévy, 1948; Pitman, 1975] If $M(t) = \max_{0\le s\le t} W(s)$, then the (two dimensional) processes $\Big(\big(M(t) - W(t),\ M(t)\big),\ t \ge 0\Big)$ and $\Big(\big(|W(t)|,\ L^0(t)\big),\ t \ge 0\Big)$ have the same distribution (in the space of continuous functions). In particular, $M - W$ is a Markov process. Moreover, if $\boldsymbol{\rho} = \boldsymbol{\rho}(t),\ t \ge 0$ is the three-dimensional Bessel process starting at 0 [that is, $\boldsymbol{\rho}(t) = \sqrt{W_1^2(t) + W_2^2(t) + W_3^2(t)}$ for iid standard Brownian motions $W_j$] and $\boldsymbol{J}(t) = \inf_{s\ge t} \boldsymbol{\rho}(s)$, then the two-dimensional processes $\Big(\big(2M(t) - W(t),\ M(t)\big),\ t \ge 0\Big)$ and $\Big(\big(\boldsymbol{\rho}(t),\ \boldsymbol{J}(t)\big),\ t \ge 0\Big)$ have the same distribution. In particular, $2M - W$ is a Markov process.

2 [Engelbert-Schmidt 0-1 Law, 1981] If $f = f(x),\ x \in \mathbb{R}$, is a non-negative Borel-measurable function, then the following three statements are equivalent:

$$\mathbb{P}\left(\int_0^t f\big(W(s)\big)\, ds < \infty,\ 0 < t < \infty\right) > 0, \quad \mathbb{P}\left(\int_0^t f\big(W(s)\big)\, ds < \infty,\ 0 < t < \infty\right) = 1,$$

$$\int_a^b f(x)\, dx < \infty, \qquad -\infty < a < b < +\infty.$$

3 [H.F. Trotter, 1958; E. Perkins, 1981] The function $(t, x) \mapsto L^x(t)$ is jointly Hölder $\frac{1}{2}$-.

4 [Ray-Knight Theorems, 1963] The goal is to fix the time variable $t$ of $L = L^x(t)$ at a suitable *stopping time* and consider the result as a function of the space variable $x$. Accordingly, for $a > 0$, consider the stopping times $\tau_a = \min\{t > 0 : W(t) = a\}$, $\sigma_a = \min\{t > 0 : L^0(t) = a\}$, and let $R(t) = \big(W_1(t), W_2(t)\big)$ be a two-dimensional standard Brownian motion independent of $W$. Then

   (a) The processes $\big(L^{a-t}(\tau_a),\ t \in [0, a]\big)$ and $\big(|R(t)|^2,\ t \in [0, a]\big)$ have the same distribution;

   (b) The processes $\big(L^t(\sigma_a) + W_1^2(t),\ t \ge 0\big)$ and $\Big(\big(W_1(t) + \sqrt{a}\big)^2,\ t \ge 0\Big)$ have the same distribution. In particular, $t \mapsto L^t(\sigma_a)$ is the squared Bessel process of order 0 starting at $a$.