Chapter 3

Hermite polynomials, spacings and limit distributions for the Gaussian ensembles

——from 'An Introduction to Random Matrices', Anderson-Guionnet-Zeitouni

In this chapter, the analysis of asymptotics for the joint eigenvalue distribution is presented for the Gaussian ensembles: the GOE, GUE and GSE. As it turns out, the analysis takes a particularly simple form for the GUE, because then the process of eigenvalues is a determinantal process. (We postpone to Section 4.2 a discussion of general determinantal processes, opting to present here all computations with bare hands.) In keeping with our goal of making this chapter accessible with minimal background, in most of this chapter we consider the GUE, and discuss the other Gaussian ensembles in Section 3.9. Generalizations to other ensembles, refinements and other extensions are discussed in Chapter 4 and in the bibliographical notes.
Recall some useful definitions of Gaussian ensembles given in Chapter 2:

Let $\{\xi_{i,j}, \eta_{i,j}\}_{i,j=1}^{\infty}$ be an i.i.d. family of real mean 0 variance 1 Gaussian random variables. We define

$$P_2^{(1)}, P_3^{(1)}, ...$$

to be the laws of the random matrices

$$\begin{bmatrix} \sqrt{2}\xi_{1,1} & \xi_{1,2} \\ \xi_{1,2} & \sqrt{2}\xi_{2,2} \end{bmatrix} \in \mathscr{H}_2^{(1)}, \qquad \begin{bmatrix} \sqrt{2}\xi_{1,1} & \xi_{1,2} & \xi_{1,3} \\ \xi_{1,2} & \sqrt{2}\xi_{2,2} & \xi_{2,3} \\ \xi_{1,3} & \xi_{2,3} & \sqrt{2}\xi_{3,3} \end{bmatrix} \in \mathscr{H}_3^{(1)}, \ldots$$

respectively. We define

$$P_2^{(2)}, P_3^{(2)}, ...$$

to be the laws of the random matrices

$$\begin{bmatrix} \xi_{1,1} & \frac{\xi_{1,2}+i\eta_{1,2}}{\sqrt{2}} \\ \frac{\xi_{1,2}-i\eta_{1,2}}{\sqrt{2}} & \xi_{2,2} \end{bmatrix} \in \mathscr{H}_2^{(2)}, \qquad \begin{bmatrix} \xi_{1,1} & \frac{\xi_{1,2}+i\eta_{1,2}}{\sqrt{2}} & \frac{\xi_{1,3}+i\eta_{1,3}}{\sqrt{2}} \\ \frac{\xi_{1,2}-i\eta_{1,2}}{\sqrt{2}} & \xi_{2,2} & \frac{\xi_{2,3}+i\eta_{2,3}}{\sqrt{2}} \\ \frac{\xi_{1,3}-i\eta_{1,3}}{\sqrt{2}} & \frac{\xi_{2,3}-i\eta_{2,3}}{\sqrt{2}} & \xi_{3,3} \end{bmatrix} \in \mathscr{H}_3^{(2)}, \ldots$$

respectively. A random matrix $X \in \mathscr{H}_N^{(\beta)}$ with law $P_N^{(\beta)}$ is said to belong to the Gaussian orthogonal ensemble (GOE) or the Gaussian unitary ensemble (GUE) according as $\beta = 1$ or $\beta = 2$, respectively. (We often write GOE(N) and GUE(N) when an emphasis on the dimension is needed.) The theory of Wigner matrices developed in previous sections of this book applies here. In particular, for fixed β, given for each N a random matrix $X(N) \in \mathscr{H}_N^{(\beta)}$ with law $P_N^{(\beta)}$, the empirical distribution of the eigenvalues of $X_N := X(N)/\sqrt{N}$ tends to the semicircle law of mean 0 and variance 1.

3.1 Summary of main results: spacing distributions in the bulk and edge of the spectrum for the Gaussian ensembles

We recall that the N eigenvalues of the GUE/GOE/GSE are spread out on an interval of width roughly equal to $4\sqrt{N}$, and hence the spacing between adjacent eigenvalues is expected to be of order $1/\sqrt{N}$.

3.1.1 Limit results for the GUE

Theorem 3.1.1 (Gaudin - Mehta). *For any compact set $A \subset \mathbb{R}$,*

$$\lim_{N\to\infty} P[\sqrt{N}\lambda_1^N, \dots, \sqrt{N}\lambda_N^N \notin A]$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \cdots \int_A det_{i,j=1}^k K_{sine}(x_i, x_j) \Pi_{j=1}^k dx_j, \tag{3.1}$$

where

$$K_{sine}(x,y) = \begin{cases} \frac{1}{\pi}\frac{sin(x-y)}{x-y}, & x \neq y, \\ \frac{1}{\pi}, & x = y. \end{cases}$$

Theorem 3.1.2 (Jimbo - Miwa - Mori - Sato). *One has*

$$\lim_{N\to\infty} P[\sqrt{N}\lambda_1^N, \dots, \sqrt{N}\lambda_N^N \notin (-t/2, t/2)] = 1 - F(t),$$

with

$$1 - F(t) = \exp\left(\int_0^t \frac{\sigma(x)}{x} dx\right) \qquad \text{for } t \geq 0,$$

with σ the solution of

$$(t\sigma'')^2 + 4(t\sigma' - \sigma)(t\sigma' - \sigma + (\sigma')^2) = 0,$$

so that

$$\sigma = -\frac{t}{\pi} - \frac{t^2}{\pi^2} - \frac{t^3}{\pi^3} + O(t^4) \qquad \text{as } t \downarrow 0.$$

The differential equation satisfied by σ is the σ -form of Painleve V. Note that Jimbo-Miwa-Mori-Sato implies that $F(t) \to 0$ as $t \to 0$. Additional analysis yields that also $F(t) \to 1$ as $t \to \infty$, showing that F is the distribution function of a probability distribution on $\mathbb{R}_+$.
We now turn our attention to the edge of the spectrum.

Definition 3.1.3. The *Airy* function is defined by the formula

$$Ai(x) = \frac{1}{2\pi i} \int_C e^{\zeta^3/3 - x\zeta} d\zeta$$

where C is the contour in the ζ-plane consisting of the ray joining $e^{-\pi i/3}\infty$ to the origin plus the ray joining the origin to $e^{-\pi i/3}\infty$.

The *Airy kernel* is defined by

$$K_{Airy}(x,y) = A(x,y) := \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y},$$

where the value for $x = y$ is determined by continuity.
By differentiating under the integral and then integrating by parts, it follows that $Ai(x)$, for $x \in \mathbb{R}$, satisfies the *Airy equation*:

$$\frac{d^2y}{dx^2} - xy = 0$$

The fundamental result concerning the eigenvalues of the GUE at the edge of the spectrum:

Theorem 3.1.4. *For all* $-\infty < t \leq T \leq \infty$,

$$\lim_{N\to\infty} P\left[N^{2/3}\left(\frac{\lambda_i^N}{\sqrt{N}} - 2\right) \notin [t,T], i = 1,\dots,N\right]$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_t^T \dots \int_t^T det_{i,j=1}^k A(x_i,x_j) \Pi_{j=1}^k dx_j, \tag{3.2}$$

where A *is the Airy kernel. In particular,*

$$\lim_{N\to\infty} P\left[N^{2/3}\left(\frac{\lambda_N^N}{\sqrt{N}} - 2\right) \leq t\right]$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_t^\infty \dots \int_t^\infty det_{i,j=1}^k A(x_i,x_j) \Pi_{j=1}^k dx_j, =: F_2(t). \tag{3.3}$$

Note that the statement of this theorem does not ensure that F_2 is a distribution function (and in particular, does not ensure that $F_2(-\infty) = 0$), since it only implies the vague convergence, not the weak convergence, of the random variables $\lambda_N^N/\sqrt{N} - 2$. The latter convergence, as well as a representation of F_2, are contained in the following.

Theorem 3.1.5 (Tracy-Widom). *The function* $F_2(\cdot)$ *is a distribution function that admits the representation*

$$F_2(t) = \exp\left(-\int_t^\infty (x-t)q(x)^2 dx\right),$$

where q *satisfies*

$$q = tq + 2q^3, q(t) \sim Ai(t) \text{ as } t \to \infty \tag{3.4}$$

The function $F_2(\cdot)$ *is the TracyWidom distribution. Equation (3.4) is the Painleve II equation.*

3.1.2 Generalizations: limit formulas for the GOE and GSE

For $\beta = 1,2,4$, let $\lambda^{(\beta,n)} = (\lambda_1^{(\beta,n)}, \dots, \lambda_n^{(\beta,n)})$ be a random vector in $\mathbb{R}$ with the law $\mathscr{P}_n^{(\beta)}$, see (2.5.6) in Chapter 2, possessing a density with respect to Lebesgue measure proportional to $|\Delta(x)|^\beta e^{-\beta|x|^2/4}$. (Thus, $\beta = 1$ corresponds to the GOE, $\beta = 2$ to the GUE and $\beta = 4$ to the GSE.) Consider the limits

$$\begin{aligned} 1 - F_{\beta,bulk}(t) &= \lim_{n\to\infty} P(\{\sqrt{n}\lambda^{(\beta,n)}\} \cap (-t/2, t/2) = \emptyset), \text{ for } t > 0, \\ F_{\beta,edge}(t) &= \lim_{n\to\infty} P(\{n^{1/6}(\lambda^{(\beta,n)} - 2\sqrt{n})\} \cap (t,\infty) = \emptyset), \text{ for all real } t. \end{aligned}$$

The existence of these limits for $\beta = 2$ follows from Theorems 3.1.2 and 3.1.4, together with Theorem 3.1.5. Further, from Lemma 3.6.6 below, we also have

$$1-F_{2,bulk}(t)=\exp\left(-\frac{t}{\pi}-\int_0^t (t-x)r(x)^2dx\right),$$

where

$$t^2((tr)''+(tr))^2=4(tr)^2((tr)^2+((tr)')^2), \qquad r(t)=\frac{1}{\pi}+\frac{t}{\pi^2}+O_{t\downarrow 0}(t^2).$$

The following is the main result of the analysis of spacings for the GOE and GSE.

Theorem 3.1.6. *The limits* $1F_{\beta,bulk}(\beta=1,4)$ *exist and are as follows:*

$$\frac{1-F_{1,bulk}(t)}{\sqrt{1-F_{2,bulk}(t)}} = \exp\left(-\frac{1}{2}\int_0^t r(x)dx\right), \tag{3.5}$$

$$\frac{1-F_{4,bulk}(t/2)}{\sqrt{1-F_{2,bulk}(t)}} = \cosh\left(-\frac{1}{2}\int_0^t r(x)dx\right). \tag{3.6}$$

Theorem 3.1.7. *The limits* $1F_{\beta,edge}(\beta=1,4)$ *exist and are as follows:*

$$\frac{F_{1,edge}(t)}{\sqrt{F_{2,edge}(t)}} = \exp\left(-\frac{1}{2}\int_t^\infty r(x)dx\right), \tag{3.7}$$

$$\frac{F_{4,edge}(t/2^{2/3})}{\sqrt{F_{2,edge}(t)}} = \cosh\left(-\frac{1}{2}\int_t^\infty r(x)dx\right). \tag{3.8}$$

3.2 Hermite polynomials and the GUE

3.2.1 The GUE and determinantal laws

Definition 3.2.1. (a) The nth Hermite polynomial $H_n(x)$ is defined as

$$H_n(x):=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}.$$

(b) The nth normalized oscillator wave-function is the function

$$\psi_n(x)=\frac{e^{-x^2/4}H_n(x)}{\sqrt{\sqrt{2\pi}n!}}.$$

Set

$$K^{(N)}(x,y)=\sum_{k=0}^{N-1}\psi_k(x)\psi_k(y).$$

Lemma 3.2.4. For any mearable subset A of $\mathbb{R}$,

$$P_N^{(2)}(\cap_{i=1}^N\{\lambda_i\in A\})=1+\sum_{k=1}^\infty\frac{(-1)^k}{k!}\int_{A^c}\cdots\int_{A^c}\det_{i,j=1}^k K^{(N)}(x_i,x_j)\Pi_{i=1}^k dx_i.$$

3.2.2 Properties of the Hermite polynomials and oscillator wave-functions

3.3 The semicircle law revisited

3.3.1 Calculation of moments of L_N

3.3.2 The Harer - Zagier recursion and Ledoux's argument

This section provides the proof of the following lemma.

Lemma 3.3.2 (Ledoux's bound). there exist positive constants c' and C' suchh that

$$P\left(\frac{\lambda_N^N}{2\sqrt{N}} \geq e^{N^{-2/3}\epsilon}\right) \leq C'e^{-c'\epsilon},$$

for all $N \geq 1$ and $\epsilon > 0$.

Roughly speaking, the last inequality says that fluctuations of the rescaled top eigenvalue $\tilde{\lambda}_N^N := \lambda_N^N/2\sqrt{N} - 1$ above 0 are of order of magnitude $N^{-2/3}$. This is an *a priori* indication that the random variables $N^{2/3}\tilde{\lambda}_N^N$ converge in distribution, as stated in Theorems 3.1.4 and 3.1.5. In fact, this inequality is going to play a role in the proof of Theorem 3.1.4, (see Subsection 3.7.1).

3.4 Quick introduction to Fredholm determinants

3.4.1 The setting, fundamental estimates and definition of the Fredholm determinant

Let X be a locally compact Polish space, with $\mathscr{B}_X$ denoting its Borel σ-algebra. Let ν be a complex-valued measure on $(X, \mathscr{B}_X)$, such that

$$||\nu||_1 = \int_X |\nu(dx)| < \infty.$$

(In many applications, $X = \mathbb{R}$, and ν will be a scalar multiple of the Lebesgue measure on a bounded interval).

Definition 3.4.1. A *kernel* is a Borel measurable, complex-valued function $K(x, y)$ defined on $X \times X$ such that

$$||K|| := \sup_{(x,y)\in X\times X} |K(x, y)| < \infty.$$

The trace of a kernel $K(x, y)$ (with respect to ν) is

$$tr(K) = \int K(x, x)d\nu(x).$$

Define the Fredholm determinant associated with a kernel $K(x, y)$. For $n > 0$, put

$$\Delta_n = \Delta_n(K, \nu) = \int \cdots \int \det_{i,j=1}^{n} K(\xi_i, \xi_j)d\nu(\xi_1)\cdots d\nu(\xi_n),$$

setting $\Delta_0 = \Delta_0(K, n) = 1$.

Definition 3.4.3. The *Fredholm determinant* associated with the kernel K is defined as

$$\Delta(K) = \Delta(K, \nu) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\Delta_n(K, \nu).$$

The determinants $\Delta(K)$ inherit good continuity properties with respect to the $||\cdot||$norm.

Lemma 3.4.5. For any two kernels $K(x, y)$ and $L(x, y)$ we have

$$|\Delta(K) - \Delta(L)| \leq \left(\sum_{n=1}^{\infty} \frac{n^{1+n/2}||\nu||_1^n \cdot \max(||K||, ||L||)^{n-1}}{n!}\right) \cdot ||K - L||.$$

3.4.2 Definition of the Fredholm adjugant, Fredholm resolvent and a fundamental identity

3.5 Gap probabilities at 0 and proof of Theorem 3.1.1

Set

$$S^{(n)}(x,y) = \frac{1}{\sqrt{n}} K^{(n)}\left(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}}\right).$$

Lemma 3.5.1. With the above notation, it holds that

$$\lim_{n\to\infty} S^{(n)}(x,y) = \frac{1}{\pi}\frac{\sin(x-y)}{x-y},$$

uniformly on each bounded subset of the (x,y)-plane.

Proof of Theorem 3.1.1. Recall that by Lemma 3.2.4,

$$\begin{aligned} & P[\sqrt{n}\lambda_1^{(n)}, \dots, \sqrt{n}\lambda_n^{(n)} \notin A] \\ = \;& 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\sqrt{n}^{-1}A} \cdots \int_{\sqrt{n}^{-1}A} \det_{i,j=1}^{k} K^{(n)}(x_i,x_j) \Pi_{j=1}^k dx_j, \\ = \;& 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{A} \cdots \int_{A} \det_{i,j=1}^{k} S^{(n)}(x_i,x_j) \Pi_{j=1}^k dx_j. \end{aligned}$$

(The scaling of Lebesgues measure in the last equality explains the appearance of the scaling by $1/\sqrt{n}$ in the definition of $S^{(n)}(x,y)$.) Lemma 3.5.1 together with Lemma 3.4.5 complete the proof of the theorem. □

The proof of Lemma 3.5.1 takes up the rest of this section.

3.5.1 The method of Laplace

3.5.2 Evaluation of the scaling limit: proof of Lemma 3.5.1

3.5.3 A complement: determinantal relations