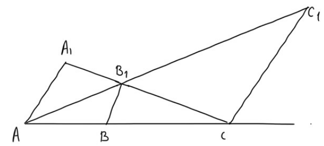
The Ladder Theorem and Beyond



If $AA_1 \parallel BB_1 \parallel CC_1$

Then $\frac{1}{|BB_1|} = \frac{1}{|AA_1|} + \frac{1}{|CC_1|}$

Proof by similar triangles,

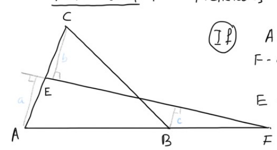
$$\frac{|BB_1|}{|AA_1|} = \frac{|BC|}{|AC|} \; ; \; \frac{|BB_1|}{|CC_1|} = \frac{|AB|}{|AC|}$$

$$\Downarrow$$

$$|BB_1| \cdot \left(\frac{1}{|CC_1|} + \frac{1}{|AA_1|} \right) = \frac{|AB| + |BC|}{|AC|} = 1.$$

$\blacksquare$

Next step: Menelaus (of Alexandria, ~70–140 AD)



If ABC is a triangle
F – a point on the extension of AB
E – a point on AC

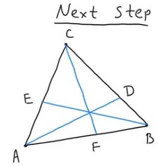
Then

$$\frac{|FA|}{|FB|} \cdot \frac{|DB|}{|DC|} \cdot \frac{|EC|}{|EA|} = 1$$

Proof $\underbrace{\frac{|FA|}{|FB|}}_{=\frac{a}{b}} \cdot \underbrace{\frac{|DB|}{|DC|}}_{=\frac{b}{c}} \cdot \underbrace{\frac{|EC|}{|EA|}}_{=\frac{c}{a}}$

$\blacksquare$

Next Step: Ceva (Giovanni Ceva (1647–1734))



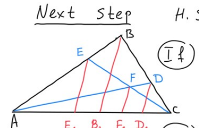
If ABC is a triangle
O – a point inside

Then $\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$

Proof
1) Menelaus on ACF, line EB
2) Menelaus on BCF, line AD

$\blacksquare$

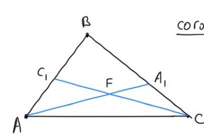
Next Step H. Stengel (2002)



If ABC is a triangle
F – a point inside
$EE_1 \parallel BB_1 \parallel FF_1 \parallel DD_1$

Then $\frac{1}{|EE_1|} + \frac{1}{|DD_1|} = \frac{1}{|BB_1|} + \frac{1}{|FF_1|}$

Proof start with right triangle
[$\widehat{BAC} = \frac{\pi}{2}$] and $E_1 = A$.
Then it is the Ladder theorem.



corollary

For every triangle ABC
and a point F inside,

$$\frac{1}{\text{area of } ABC} + \frac{1}{\text{area of } AFC} = \frac{1}{\text{area of } AC_1C} + \frac{1}{\text{area of } AA_1C}$$